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Magnetism and Electromagnetic Waves

C C 3

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For the course content, see the syllabus.

2nd Semester

L: Lorentz force: $\vec{F}_E = q\vec{E}$, $\vec{F}_B = q(\vec{v}\times\vec{B})$

$$\vec{F} = \vec{F}_E + \vec{F}_B \Rightarrow \boxed{\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})}$$

(Lorentz Force)

Force on a current carrying conductor:

No. of e^- in elementary len. dl, $= (NAdl)$

N = charge den.

Magnetic force, $d\vec{F}_B = NAdl\,e\,(\vec{v}\times\vec{B})$

$$\Rightarrow d\vec{F}_B = (NAe\vec{v}\,dl\times\vec{B})$$

$$= I\,d\vec{l}\times\vec{B}$$

$$\vec{F}_B = \int d\vec{F}_B = I\int d\vec{l}\times\vec{B}$$

$$\Rightarrow \boxed{\vec{F}_B = I(\vec{l}\times\vec{B})}$$

For closed loop, $\vec{l} = 0 \Rightarrow \underline{\vec{F}_B = 0}$

Monopole never exists for $\vec{B}$, so, $\boxed{\vec{\nabla}\cdot\vec{B} = 0}$

$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \leftrightarrow$ Ampere's Law.

from Maxwell's eq^n $\rightarrow$ $\boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}}$ $\boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}}$

Now, $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \Rightarrow \vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \mu_0(\vec{\nabla}\cdot\vec{J})$

$$\Rightarrow \underline{\vec{\nabla}\cdot\vec{J} = 0}$$

Eq^n of continuity, $\boxed{\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0}$

$$\Rightarrow \frac{\partial\rho}{\partial t} = 0$$

$\rightarrow \rho =$ const. (t independent)

(Steady current)

$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \int(\vec{\nabla}\cdot\vec{B})\,dV = 0$

$\Rightarrow \oint \vec{B}\cdot d\vec{S} = 0$

$\downarrow$

(Magnetic flux) ~~= 0~~

Magnetic flux: $\phi_B = \int \vec{B}\cdot d\vec{S}$

It's solenoidal field

Ampere's circuital law: $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$

So, $\int_c (\vec{\nabla}\times\vec{B})\cdot d\vec{S} = \mu_0 \int_S \vec{J}\cdot d\vec{S}$

$$\Rightarrow \boxed{\oint \vec{B}\cdot d\vec{l} = \mu_0 I}$$

~~Magnetic Vector Potential:~~

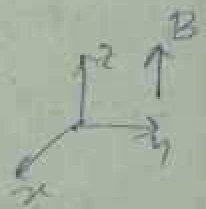
L Unit of B (magnetic induction) $\rightarrow \frac{\text{Volt s}}{m^2}, \frac{W}{m^2}$ (W $\rightarrow$ weber), T (Tesla) (SI)

$1\,T = 10^4$ Gauss $\quad [B] = MT^{-2}I^{-1}$

$\uparrow$ SI $\quad\quad \uparrow$ CGS

Motion of a charged particle in a uniform magnetic field ($\vec{B}$):

$$\vec{F}_B = m\frac{d\vec{v}}{dt} = q(\vec{v}\times\vec{B}) \quad -(1)$$



$$\Rightarrow m\frac{d}{dt}(v_x\hat{i} + v_y\hat{j} + v_z\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & v_z \\ 0 & 0 & B \end{vmatrix} \cdot q$$

$$= (v_y B\hat{i} - v_x B\hat{j})\cdot q$$

$$\Rightarrow m\frac{dv_x}{dt} = qv_yB \quad -(2), \quad m\frac{dv_y}{dt} = -v_xBq \quad -(3)$$

$$\Rightarrow m\frac{dv_z}{dt} = 0 \Rightarrow v_z = \text{const} = v_{\parallel} \quad -(4)$$

So, vel. $\parallel$ to $\vec{B}$ is const. ($\parallel$ to $\vec{B}$)

So, $v_z = \frac{dz}{dt} = v_{\parallel} \rightarrow \boxed{z = v_{\parallel} t + z_0}$ (5)

(z at $t=0$)

(2), (3) $\rightarrow$ $\frac{dv_x}{dt} = \omega v_y$, $\frac{dv_y}{dt} = -\omega v_x$ $\quad \boxed{\omega = \frac{qB}{m}}$ cyclotron freq.

So, $\frac{dv_x}{dt} + j\frac{dv_y}{dt} = \omega v_y - j\omega v_x$

$\Rightarrow \frac{d}{dt}(v_x + jv_y) = -j\omega\left(v_x - \frac{1}{j}v_y\right)$

$\Rightarrow \frac{d}{dt}(v_x + jv_y) = -j\omega(v_x + jv_y)$

$\Rightarrow \ln(v_x + jv_y) = -j\omega t$ (integrating) $+ \underbrace{\ln c}_{\text{const}}$

$\Rightarrow v_x + jv_y = c e^{-j\omega t}$ — (6)

$\Rightarrow \frac{d}{dt}(x + jy) = c\, e^{-j\omega t}$

$\Rightarrow x + jy = \frac{c}{-j\omega} e^{-j\omega t} + c_2$

$\Rightarrow x + jy = c_1 e^{-j\omega t} + c_2$

Let, $c_1 = Ae^{-j\alpha}$, $c_2 = x_0 + jy_0 \rightarrow$ consts

So, $x + jy = Ae^{-j(\omega t + \alpha)} + x_0 + jy_0$

$\Rightarrow x + jy = (A\cos(\omega t + \alpha) + x_0) + j(A\sin(\omega t + \alpha) + y_0)$

$\Rightarrow x = A\cos(\omega t + \alpha) + x_0$

& $y = -A\sin(\omega t + \alpha) + y_0$ — (7)

This is the equation for particle trajectory.

So, $(x - x_0)^2 = A^2\cos^2(\omega t + \alpha)$

$(y - y_0)^2 = A^2\sin^2(\omega t + \alpha)$

$$\boxed{(x - x_0)^2 + (y - y_0)^2 = A^2} \quad \text{— (8)}$$

circle in xy plane, $C = (x_0, y_0)$, $R = A$

(If $v_{\parallel}=0$, the particle describes a circular path in the plane $\perp$ to $\vec{B}$.)

Now, (7) $\to$ $\dot{x} = -\omega A \sin(\omega t+\alpha)$

$\dot{y} = -\omega A \cos(\omega t+\alpha)$

So, $\dot{x}^2+\dot{y}^2 = \omega^2 A^2$

$\Rightarrow v_\perp^2 = \omega^2 A^2 \Rightarrow A = \frac{v_\perp}{\omega}$

$\Rightarrow A = \frac{v_\perp}{qB/m}$

(9) $\Rightarrow \boxed{A = \frac{m v_\perp}{qB}}$ It is radius of gyration.

Another method, ~~for $v_{\parallel}=0$~~,

$\frac{m v_\perp^2}{r} = qvB\sin 90^\circ \Rightarrow r = \frac{m v_\perp}{qB}$

for circular motion.

The momentum of particle,

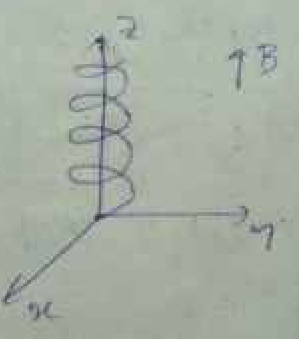
$p = mv = qBr \Rightarrow p = qBr$

Time period, $T = \frac{2\pi}{\omega}$

$\Rightarrow T = \frac{2\pi}{qB/m} \Rightarrow \boxed{T = \frac{2\pi m}{qB}}$

If $v_{\parallel} \neq 0$, particles moves parallel to $\vec{B}$ with const. speed $v_{\parallel}$ and gyrates about $\vec{B}$ in time T. ~~The par~~ So, the particle moves in a helix with $\vec{B}$ as an axis.

The dir. of gyration oppose the dir. of $\vec{B}$

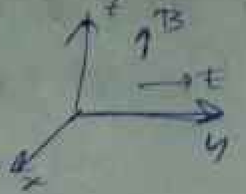


Motion of a charged particle in crossed $\vec{E}$ and $\vec{B}$. :

Particle at origin in rest at $t=0$

$$m\frac{d\vec{v}}{dt} = q\vec{E} + q(\vec{v}\times\vec{B}) \quad -(1)$$

Let $\underline{\vec{E}\perp\vec{B}}$, $\vec{E} = E\hat{j}$, $\vec{B} = B\hat{k}$



$$m\frac{d\vec{v}}{dt} = qE\hat{j} + q\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & v_z \\ 0 & 0 & B \end{vmatrix}$$

$$\Rightarrow m\frac{d}{dt}(v_x\hat{i} + v_y\hat{j} + v_z\hat{k}) = \hat{i}(qv_yB) + \hat{j}(qE - qv_xB)$$

$$\Rightarrow m\frac{dv_x}{dt} = qv_yB,\ m\frac{dv_y}{dt} = qE - qv_xB,\ m\frac{dv_z}{dt} = 0 \quad (2)$$

$$\Rightarrow \frac{dv_x}{dt} = \omega v_y,\ \frac{dv_y}{dt} = \frac{qE}{m} - \omega v_x,\ \frac{dv_z}{dt} = 0$$

cyclotron freq., $\omega = qB/m$.

$$\Rightarrow \left(\frac{dv_x}{dt} = \omega v_y,\ \frac{dv_y}{dt} = a_y - \omega v_x,\ \frac{dv_z}{dt} = 0\right) \quad (2)$$

Acc^n due to $\vec{E}$ is $a_y = \frac{qE}{m}$.

$$\frac{d^2v_x}{dt^2} = \omega\frac{dv_y}{dt} = \omega a_y - \omega^2 v_x \quad -(3)$$

$$\frac{d^2v_y}{dt^2} = \frac{da_y}{dt}^{0} - \omega\frac{dv_x}{dt} = -\omega^2 v_y \quad -(4)$$

$$\Rightarrow \frac{d^2v_y}{dt^2} + \omega^2 v_y = 0$$

$$\Rightarrow v_y = A_1\sin\omega t + A_2\cos\omega t$$

At, $t=0$, $v_y = 0 \Rightarrow A_2 = 0$. So, $v_y = A_1\sin\omega t$ $\quad -(5)$

$$\int dy = \int A_1\sin\omega t\,dt$$

$$\Rightarrow y = -\frac{A_1}{\omega}\cos\omega t + c_1$$

At $t=0$, $y=0$, $\Rightarrow -\frac{A_1}{\omega} + c_1 = 0$

So, $y = -\frac{A_1}{\omega}\cos\omega t + \frac{A_1}{\omega}$

$$\Rightarrow y = \frac{A_1}{\omega}(1-\cos\omega t) \quad -(6)$$

(2) (5) $\rightarrow \frac{dv_x}{dt} = \omega A_1 \sin\omega t$

$\Rightarrow v_x = -\frac{\omega A_1}{\omega} \cos\omega t + C_2$

At $t=0$, $v_x = 0 \rightarrow 0 = -A_1 + C_2$

So, $v_x = -A_1 \cos\omega t + A_1$

$\Rightarrow v_x = A_1(1-\cos\omega t)$ — (7)

$\Rightarrow \int_0^x dx = A_1 \int_0^t (1-\cos\omega t)\,dt$

$\Rightarrow x = A_1\left[t - \left(\frac{\sin\omega t}{\omega}\right)_0^t\right]$

$\Rightarrow x = A_1\left(t - \frac{\sin\omega t}{\omega}\right)$

$\Rightarrow x = \frac{A_1}{\omega}(\omega t - \sin\omega t)$ — (8)

(6) $\Rightarrow v_y = A_1 \sin\omega t \Rightarrow \frac{dv_y}{dt} = \omega A_1 \cos\omega t$

(2) $\rightarrow a_y - \omega v_x = \frac{dv_y}{dt} = \omega A_1 \cos\omega t$

$\Rightarrow A_1 = \frac{a_y - \omega v_x}{\omega \cos\omega t}$

At $t=0$, $v_x = 0 \longrightarrow A_1 = \frac{a_y - 0}{\omega}$

$\Rightarrow A_1 = \frac{a_y}{\omega} = \frac{qE/m}{qB/m}$

$y = \frac{E}{B}\sin\omega t$

$\Rightarrow A_1 = \frac{E}{B}$ — (9)

If $v_x \neq 0$ at $t=0 \rightarrow A_1 = \frac{a_y}{\omega} - v_{x_0}$

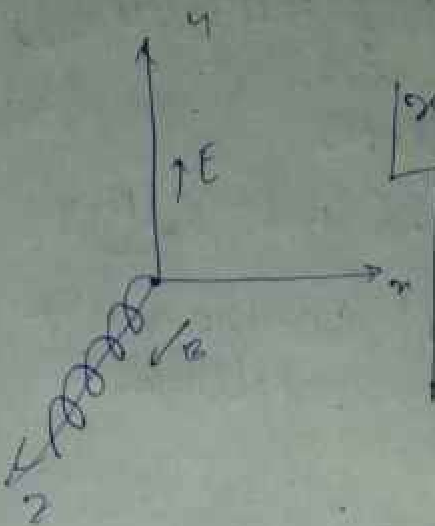
Here, $\Rightarrow A_1 = \frac{E}{B} - v_{x_0}$

$y = \left(\frac{E}{B} - v_{x_0}\right)\sin\omega t$

If, $v_{x_0} = \frac{E}{B}$, $v_y = 0 \rightarrow$ particle goes undeflected along y.

So, crossfield acts as a velocity filter for the charge particle.

(2) $v_z = \text{const}$, $dz = v_z\,dt \Rightarrow \underline{z = v_z t + z_0}$ — (10)



$$x = \frac{E}{\omega B}(\omega t - \sin\omega t)$$
$$y = \frac{E}{\omega B}(1 - \cos\omega t)$$
$$z = v_{z_0} t + z_0$$

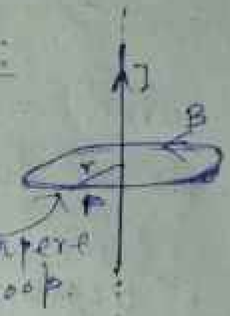
Ampere's Law: $\oint_c \vec{B}\cdot d\vec{l} = \mu_0 I$

I = total current enclosed by c.

Application:

<u>Long current carrying wire:</u>

B is same throughout the ampere loop.

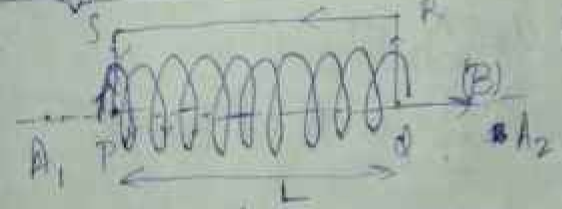


$\oint \vec{B}\cdot d\vec{l} = B \oint dl$

$= B \cdot 2\pi r$

So. $B \cdot 2\pi r = \mu_0 I \Rightarrow \boxed{B = \frac{\mu_0 I}{2\pi r}}$

<u>Long solenoid:</u>



$A_1 \to A_2$

Amperian loop → PQRS, current = I.

No. of turns = N $\quad \frac{\text{turn}}{\text{len}} \cdot n = \frac{N}{L}$.

$\oint \vec{B}\cdot d\vec{l} = \mu_0 (NI)$

PQRS $\Rightarrow \int_P^Q \vec{B}\cdot d\vec{l} + \int_Q^R \vec{B}\cdot d\vec{l} + \int_R^S \vec{B}\cdot d\vec{l}$

$+ \int_S^P \vec{B}\cdot d\vec{l} = \mu_0 N I$

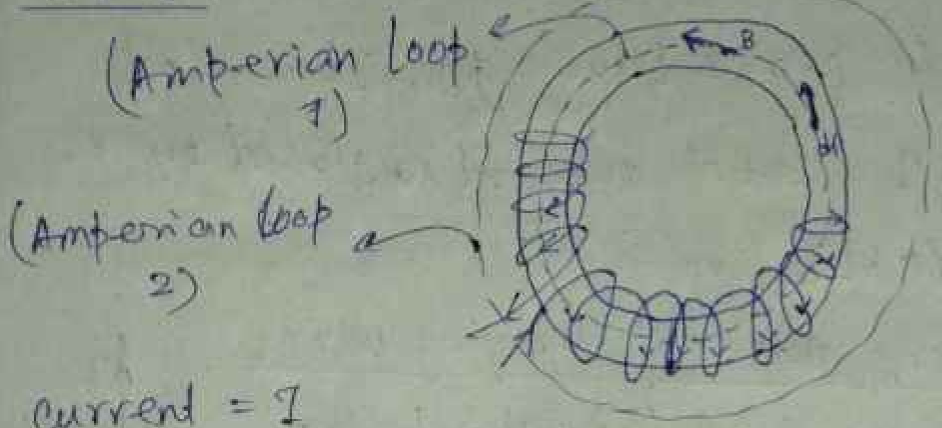
$$\Rightarrow \int_0^L B\,dl \cos 0^\circ + \int_Q^R B\,dl \cos 90^\circ + \int_R^S 0 \cdot \vec{dl} + \int_S^P B\,dl \cos 90^\circ = \mu_0 N I$$

$$\Rightarrow B \int_0^L dl = \mu_0 N I$$

$$\Rightarrow BL = \mu_0 N I \Rightarrow B = \mu_0 \left(\frac{N}{L}\right) I$$

So, $\boxed{B = \mu_0 n I}$ On the axis of solenoid.

Toroid:



current $= I$

No. of turns $= N$

$$\frac{\text{turns}}{\text{length}} = \frac{N}{2\pi R} = n$$

for loop 1 $\oint \vec{B} \cdot \vec{dl} = \mu_0 (NI)$

$$\Rightarrow \oint B\,dl \cos 0^\circ = \mu_0 N I$$

$$\Rightarrow B \cdot 2\pi R = \mu_0 N I \quad \left[\oint dl = 2\pi R\right]$$

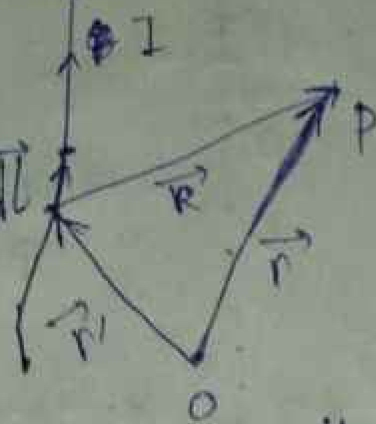
$$\Rightarrow B = \mu_0 \left(\frac{N}{2\pi R}\right) I$$

$\Rightarrow \boxed{B = \mu_0 n I}$ axis of toroid.

for loop 2, $\oint \vec{B} \cdot \vec{dl} = \mu_0 \cdot (0) \Rightarrow B = 0$

outside toroid.

Biot Savart's Law :



$\vec{R} = \vec{r} - \vec{r'}$

$\vec{B}$ at P due to $\vec{dl}$,

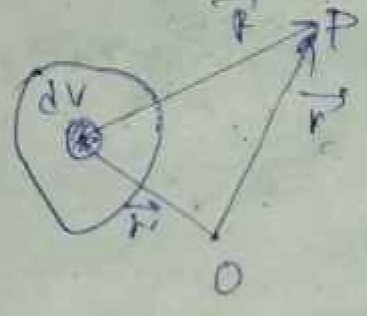
$$\boxed{d\vec{B} = \left(\frac{\mu_0}{4\pi}\right) \cdot \frac{I\, \vec{dl} \times \vec{R}}{R^3}} \quad \text{— (1)}$$

Here, $\mu_0 = 4\pi \times 10^{-7}\ N/A^2$
free space permiability

current density $= \vec{J}$

• Current element $= I\vec{dl}$

$I\vec{dl} = \vec{J} \cdot (\hat{n}\, ds \cdot \vec{dl}) = \vec{J}\, dV \rightarrow$ Volume Element



$\vec{R} = \vec{r} - \vec{r'}$

$$\boxed{d\vec{B} = \left(\frac{\mu_0}{4\pi}\right) \frac{\vec{J} \times \vec{R}\, dV}{R^3}} \quad \text{— (2)}$$

(1) $\rightarrow$ $\vec{B} = \frac{\mu_0}{4\pi} \int \frac{I\, \vec{dl} \times \vec{R}}{R^3}$, for whole length.

(2) $\rightarrow$ $\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \times \vec{R}}{R^3} dV$, for whole volume

for Surface current $\rightarrow$

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{K} \times \vec{R}}{R^3}\, ds}$$

Application:

Long St. wire:

$\vec{B}$ ... $l = a\tan\theta$

$R = a\sec\theta$. $dl = a\sec^2\theta\, d\theta$

$\vec{B} = \int \left(\frac{\mu_0}{4\pi}\right) \frac{I\,dl.\,R\sin\phi}{R^3}\,\hat{n}$

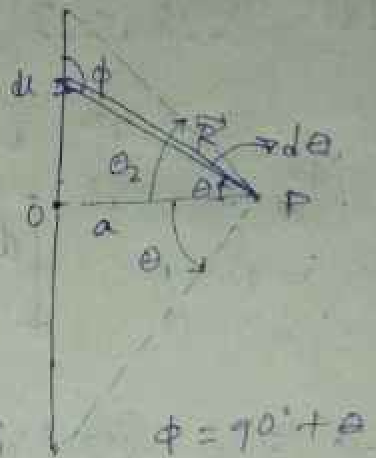
$\rightarrow B = \frac{\mu_0 I}{4\pi} \int \frac{a\sec^2\theta\cos\theta\, d\theta}{a^2\sec^2\theta}$

$= \frac{\mu_0 I}{4\pi a} \int_{-90^\circ}^{90^\circ} \cos\theta\, d\theta$

$\rightarrow \boxed{B = \frac{\mu_0 I}{2\pi a}}$

$\phi = 90° + \theta$

Straight wire:



$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I\,d\vec{l} \times \vec{R}}{R^3}$

$\rightarrow \vec{B} = \frac{\mu_0}{4\pi} \int \frac{I\,d\vec{l} \times \vec{R}}{R^3}$

$= \frac{\mu_0}{4\pi} \int \frac{I\,dl\sin\phi}{R^2}\,\hat{n}$

$= \frac{\mu_0}{4\pi} \int \frac{I\,dl\cos\theta}{R^2}\,\hat{n}$

$\phi = 90° + \theta$

$\frac{l}{a} = \tan\theta \Rightarrow dl = a\sec^2\theta\, d\theta,\ R = a\sec\theta$

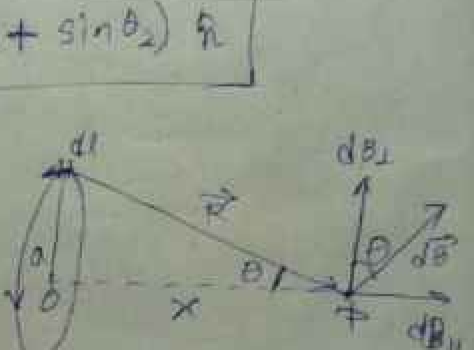
$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{I.\,a\sec^2\theta\, d\theta.\,\cos\theta}{a^2\sec^2\theta}\,\hat{n}$

$= \frac{\mu_0 I}{4\pi a} \int_{-\theta_1}^{\theta_2} \cos\theta\, d\theta\,\hat{n}$

$= \frac{\mu_0 I}{4\pi a} \left[\sin\theta\right]_{\theta_1}^{\theta_2}\,\hat{n}$

$\rightarrow \boxed{\vec{B} = \frac{\mu}{4\pi a}(\sin\theta_1 + \sin\theta_2)\,\hat{n}}$

Circular loop:



$\vec{B}$ at P for elementary length $d\vec{l}$ is,

$$d\vec{B} = \frac{\mu_0}{4\pi}\cdot\frac{I\,d\vec{l}\times\vec{R}}{R^3} = \frac{\mu_0}{4\pi}\cdot\frac{I\,dl}{R^2}$$

All $dB_\perp$ ~~cancells~~ are cancelled. $(d\vec{l}\perp\vec{R})$

$$\vec{B} = \hat{x}\int dB_\parallel = \int\frac{\mu_0}{4\pi}\frac{I\,dl}{R^2}\sin\theta\,\hat{x}$$

$$= \frac{\mu_0}{4\pi}\frac{I}{R^2}\cdot\frac{a}{R}\int_0^{2\pi a} dl\,\hat{x}$$

$$= \frac{\mu_0 I}{4\pi}\frac{a}{R^3}\cdot 2\pi a\,\hat{x} \qquad R=\sqrt{a^2+x^2}$$

$$= \cancel{\frac{\mu_0}{4\pi}}\ \frac{\mu_0 I}{4\pi}\cdot\frac{2a^2\pi}{(a^2+x^2)^{3/2}}\,\hat{x}$$

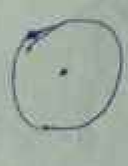
$$\Rightarrow \boxed{\vec{B} = \frac{\mu_0 I}{2}\frac{a^2}{(a^2+x^2)^{3/2}}\,\hat{x}}$$

For coil having N turns, $\vec{B}$ at point of axis is,

$$\vec{B}_{axis} = \frac{\mu_0 N I}{2}\frac{a^2}{(a^2+x^2)^{3/2}}\,\hat{x}$$

At centre of loop, $x=0$,

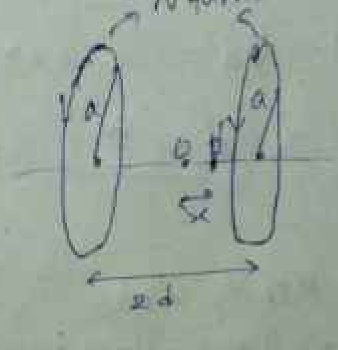
$$B_{center} = \frac{\mu_0 I}{2}\frac{a^2}{a^3}$$



$$\Rightarrow \boxed{B_{center} = \frac{\mu_0 I}{2a}}$$

(max.)

Helmoltz Coil:



At center, O,

$$B_{net} = \frac{\mu_0 N I}{2}\frac{a^2}{(a^2+d^2)^{3/2}} + \frac{\mu_0 N I}{2}\frac{a^2}{(a^2+d^2)^{3/2}}$$

$$\Rightarrow B_{net} = \mu_0 N I\,\frac{a^2}{(a^2+d^2)^{3/2}}$$

At O', $B_{net} = \frac{\mu_0 N I}{2} \frac{a^2}{(a^2+(d+x)^2)^{3/2}} +$

for max, $\frac{\mu_0 N I}{2} \frac{a^2}{(a^2+(d-x)^2)^{3/2}}$

$\frac{dB_{net}}{dx} = 0$

$\Rightarrow \frac{d}{dx}\frac{1}{(a^2+(d+x)^2)^{3/2}} + \frac{d}{dx}\frac{1}{(a^2+(d-x)^2)^{3/2}} = 0$

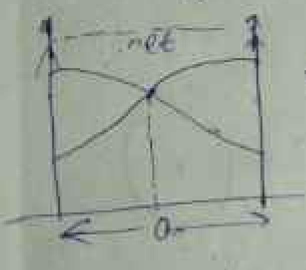
$\Rightarrow -\frac{3}{2}\frac{2(d+x)}{(a^2+(d+x)^2)^{5/2}} - \frac{3}{2}\frac{2(d-x)(-1)}{(a^2+(d-x)^2)^{5/2}} = 0$

$\Rightarrow \frac{(d+x)}{(a^2+(d+x)^2)^{5/2}} = \frac{d-x}{(a^2+(d-x)^2)^{5/2}}$

$\Rightarrow \frac{d+x}{d-x} = \left(\frac{a^2+(d+x)^2}{a^2+(d-x)^2}\right)^{5/2}$

$\Rightarrow \frac{d+x}{d-x} = \left(\frac{a^2+2dx+d^2}{a^2-2dx+d^2}\right)^{5/2}$ (Now, $x^2 < a^2+d^2$)

$\Rightarrow \frac{d+x}{d-x} = \left(\frac{1+\frac{2dx}{d^2+a^2}}{1-\frac{2dx}{d^2+a^2}}\right)^{5/2}$



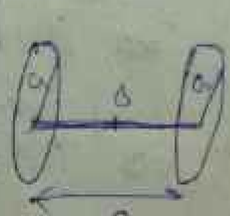
$= \frac{1+\frac{5dx}{d^2+a^2}}{1-\frac{5dx}{d^2+a^2}}$

$\Rightarrow \frac{2d}{2x} = \frac{2}{2.\ 5dx/d^2+a^2} = \frac{d^2+a^2}{5xd}$

$\Rightarrow 5d^2 = d^2+a^2 \Rightarrow 2d = a$

So, $\boxed{d = \frac{a}{2}}$ So, Magnetic field is max at O.

Hence, the configaration is,



$\boxed{B = \frac{8\mu_0 N I}{5\sqrt{5}a}}$

At O, $B_{net} = \frac{\mu_0 N I}{2} \frac{a^2}{(a^2+(\frac{a}{2})^2)^{3/2}} = \frac{8\mu_0 N I}{5\sqrt{5}\,a}$

Magnetic Vector Potential: ($\vec{A}$)

We know, $\vec{\nabla}\cdot\vec{B}=0$ (As monopole doesn't exists)

Now, div. of any curl $=0$. i.e. $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})=0$.

So, $\boxed{\vec{B}=\vec{\nabla}\times\vec{A}}$ —(1) Here, $\vec{A}$ = magnetic vector potential.

Ampere's law: $\oint \vec{B}\cdot\vec{dl} = \mu_0 I$

$\Rightarrow \int(\vec{\nabla}\times\vec{B})\cdot d\vec{S} = \mu_0 \vec{J}\cdot d\vec{S}$ ~~(2)~~

$\rightarrow \vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ — (2)

$\rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \cancel{\mu_0\vec{J}}\ \mu_0\vec{J}$

$\Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J}$ — (3)

For same $\vec{B}$ we can get diff $\vec{A}$s.

$\vec{A}$ isn't unique, we can use Gauge Transformation. $\rightarrow$ Coulomb Gauge.

$\vec{\nabla}\cdot\vec{A}=0.$

So, (3) $\longrightarrow$ $\boxed{\nabla^2\vec{A}=-\mu_0\vec{J}}$ —(4) vector Poisson equation.

Similar to, $\nabla^2\phi = -\frac{\rho}{\epsilon_0}$ in Electrostatics.

$$\phi = \frac{1}{4\pi\epsilon_0}\int\frac{\rho\, dV}{R}$$

$\phi\rightarrow 0$ for $R\rightarrow\infty$

Similarly, we can write, $\boxed{\vec{A} = \frac{\mu_0}{4\pi}\int\frac{\vec{J}\,dV}{R}}$ — (5)

Biot Savart's Law: $\vec{B} = \frac{\mu_0}{4\pi}\int\frac{(\vec{J}\times\vec{R})\,dV}{R^3}$ —(6)

Now, $\vec{\nabla}\left(\frac{1}{R}\right) = -\frac{\vec{R}}{R^3}$

$\vec{\nabla}\times\left(\frac{1}{R}\right)\vec{J} = \vec{\nabla}\left(\frac{1}{R}\right)\times\vec{J} + \frac{1}{R}\vec{\nabla}\times\vec{J}$

$\vec{J}$ = func of (x', y', z') see before Biot-savart Law

∇ is f^n of (x, y, z)

So, $\vec{\nabla} \times \vec{J} = 0$.

So, $\vec{\nabla} \times \left(\frac{1}{R}\right) \vec{J} = -\frac{\vec{R}}{R^3} \times \vec{J}$

$\Rightarrow \frac{\vec{J} \times \vec{R}}{R^3} = \vec{\nabla} \times \left(\frac{1}{R}\right) \vec{J}$

So, $\vec{B} = \frac{\mu_0}{4\pi} \int \vec{\nabla} \times \left(\frac{1}{R}\right) \vec{J} \, dV$

~~Now~~ $\boxed{\vec{B} = \vec{\nabla} \times \frac{\mu_0}{4\pi} \int \frac{\vec{J}}{R} dV}$

Comparing with, $\vec{B} = \vec{\nabla} \times \vec{A}$

$$\boxed{\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}}{R} dV}$$

In terms of line current:

$$\boxed{\vec{A} = \frac{\mu_0}{4\pi} \int \frac{I \, d\vec{l}}{R}}$$

In terms of surface current:

$$\boxed{\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{K} \, d\vec{s}}{R}}$$

<u>Biot-Savart's Law & Ampere's Law from $\vec{A}$:</u>

$\vec{A}(r)$ at point $\vec{r}(x, y, z)$ due to vol. distribution $\vec{J}(x', y', z')$ is given by

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \, dV}{R} \qquad R = |\vec{r} - \vec{r}'|, \quad dV = dx' dy' dz'$$

Now, $\vec{B} = \vec{\nabla} \times \vec{A}$

$\Rightarrow \vec{B} = \frac{\mu_0}{4\pi} \int \vec{\nabla} \times \frac{\vec{J} \, dV}{R}$

$= \frac{\mu_0}{4\pi} \left[\int \vec{\nabla}\left(\frac{1}{R}\right) \times \vec{J} \, dV + \int \frac{1}{R} \vec{\nabla} \times \vec{J} \, dV \right]$

$\vec{\nabla} \rightarrow f^n$ of (x, y, z), $\vec{J} \rightarrow f^n$ of (x', y', z')

$$\vec{\nabla} \times \vec{J} = 0$$

so, $\vec{B} = \frac{\mu_0}{4\pi}\left[\int\left(\frac{-\vec{R}}{R^3}\right) \times \vec{J}\, dV + 0\right]$

$$\Rightarrow \vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \times \vec{R}\, dV}{R^3}$$

for volume current, magnetic field,

$$\underline{d\vec{B} = \frac{\mu_0}{4\pi} \frac{\vec{J} \times \vec{R}}{R^3} dV}$$

for line current element,

$$\underline{d\vec{B} = \frac{\mu_0}{4\pi} \frac{I\, d\vec{l} \times \vec{R}}{R^3}} \rightarrow \text{Biot-Savert's Law.}$$

for surface current element,

$$\underline{d\vec{B} = \frac{\mu_0}{4\pi} \frac{\vec{K} \times \vec{R}\, ds}{R^3}}$$

Now, $\oint \vec{B} \cdot d\vec{l} = \oint (\vec{\nabla} \times \vec{A}) \cdot d\vec{l}$

$$= \int_s \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) \cdot d\vec{S}$$

so $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\underbrace{\vec{\nabla} \cdot \vec{A}}_{0 \text{ for coulomb Gauge.}}) - \nabla^2 \vec{A}$

$$= -\nabla^2 \vec{A}$$

$$= \mu_0 \vec{J}$$

So, $\oint \vec{B} \cdot d\vec{l} = \int_s \mu_0 \vec{J} \cdot d\vec{S}$

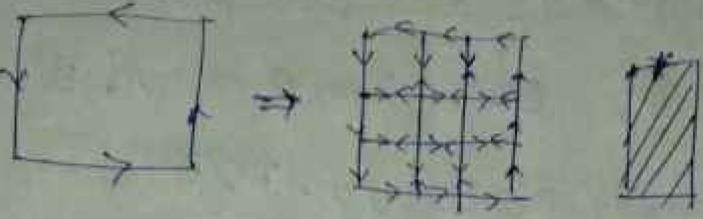
$$\Rightarrow \underline{\oint \vec{B} \cdot d\vec{l} = \mu_0 I} \rightarrow \text{Ampere's Law.}$$

1 Current loop in ext. $\vec{B}$:

force on current loop with element $i\vec{dl}$,

$d\vec{F} = i\vec{dl} \times \vec{B}$

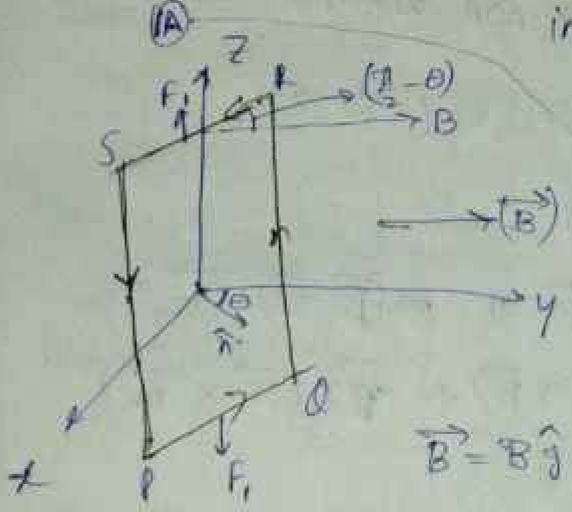
Torque on element, $d\vec{\tau} = \vec{r} \times d\vec{F} = i\,\vec{r} \times (\vec{dl} \times \vec{B})$

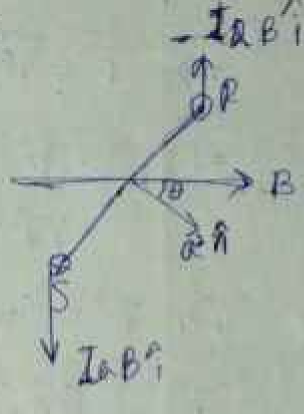


Total torque, $\vec{\tau} = i\oint_c \vec{r} \times (\vec{dl} \times \vec{B})$

If magnetic moment = $\vec{m}$,

$\vec{\tau} = \vec{m} \times \vec{B}$ (comparison with, $\vec{\tau} = \vec{p} \times \vec{E}$ in electrostatics)





$\vec{F}_{SP} = I(-a\hat{k} \times B\hat{j})$

$= IaB\hat{i}$

$\vec{F}_{QR} = I(a\hat{k} \times B\hat{j}) = -aIB\hat{i}$

Here, $\vec{F}_{net} = 0$, ~~$\vec{\tau} = \vec{r} \times \vec{F}$~~

$\vec{\tau} =$ ~~IaB~~ $IaB.\,a\sin\theta\,\hat{k}$

$\Rightarrow \vec{\tau} = Ia^2B\sin\theta\,\hat{k}$, $\vec{\tau} = \vec{m} \times \vec{B} = mB\sin\theta\,\hat{k}$

$m = Ia^2$

m = current × Area

Work done in rotating the dipole from some ref. position θ_0 to the position θ:

$$U = \int_{\theta_0}^{\theta} mB\sin\theta \, d\theta = mB(\cos\theta_0 - \cos\theta)$$

Hence, $\boxed{U = mB(\cos\theta_0 - \cos\theta)}$

For, $\theta_0 = \frac{\pi}{2}$, $U = -mB\cos\theta = -\vec{m}.\vec{B}$

$$U = -\vec{m}\cdot\vec{B} \equiv U = -\vec{p}\cdot\vec{B}$$

For a closed loop in an uniform $\vec{B}$, $\vec{F} = 0$.

Force on current loop of dipole moment $\vec{m}$ placed in a non-uniform $\vec{B}$:

$$\vec{F} = -\vec{\nabla}U = -\vec{\nabla}(-\vec{m}\cdot\vec{B})$$

$$\Rightarrow \vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})$$

$$\Rightarrow \vec{F} = (\vec{m}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{m} + \vec{m}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{m})$$

for $\vec{m} = $ (const.)

$$\vec{F} = (\vec{m}\cdot\vec{\nabla})\vec{B} + \vec{m}\times(\vec{\nabla}\times\vec{B})$$

Here, $\vec{\nabla}\times\vec{B} = 0$ means magnetic field has no sources at the position of dipole.

so, $\boxed{\vec{F} = (\vec{m}\cdot\vec{\nabla})\vec{B}}$

Similar to $\vec{F} = (\vec{P}\cdot\vec{\nabla})\vec{E}$

General result for $\vec{\tau}$:

$$d\vec{\tau} = \vec{r}\times d\vec{F} = I\,\vec{r}\times(d\vec{l}\times\vec{B})$$

Now,

$$\vec{r}\times(d\vec{l}\times\vec{B}) + d\vec{l}\times(\vec{B}\times\vec{r}) + \vec{B}\times(\vec{r}\times d\vec{l}) = 0$$

$$\Rightarrow \vec{r}\times(d\vec{l}\times\vec{B}) = -d\vec{l}\times(\vec{B}\times\vec{r}) - \vec{B}\times(\vec{r}\times d\vec{l})$$

$$d[\vec{r}\times(\vec{r}\times\vec{B})] = d\vec{r}\times(\vec{r}\times\vec{B}) + \vec{r}\times(d\vec{r}\times\vec{B}) + \vec{r}(\vec{r}\times d\vec{B}) \rightarrow 0$$

We can write, $(d\vec{r} = d\vec{l})$

$$\vec{r}\times(d\vec{l}\times\vec{B}) = d[\vec{r}\times(\vec{r}\times\vec{B})] - d\vec{r}\times(\vec{r}\times\vec{B})$$

$$2\,\vec{r}\times(d\vec{l}\times\vec{B}) = d[\vec{r}\times(\vec{r}\times\vec{B})] - \vec{B}\times(\vec{r}\times d\vec{l})$$

Hence, $d\vec{\tau} = \frac{1}{2} I \left[d[\vec{r}\times(\vec{r}\times\vec{B})] - \vec{B}\times(\vec{r}\times d\vec{l}) \right]$

Total, $\vec{\tau} = \oint d\vec{\tau}$

$$\Rightarrow \vec{\tau} = \frac{1}{2} I \oint d[\vec{r}\times(\vec{r}\times\vec{B})] - \frac{1}{2} I \oint \vec{B}\times(\vec{r}\times d\vec{l})$$

$$\Rightarrow \vec{\tau} = \frac{I}{2}\vec{r}\times(\vec{r}\times\vec{B}) - \frac{I}{2}\vec{B}\times\oint(\vec{r}\times d\vec{l})$$

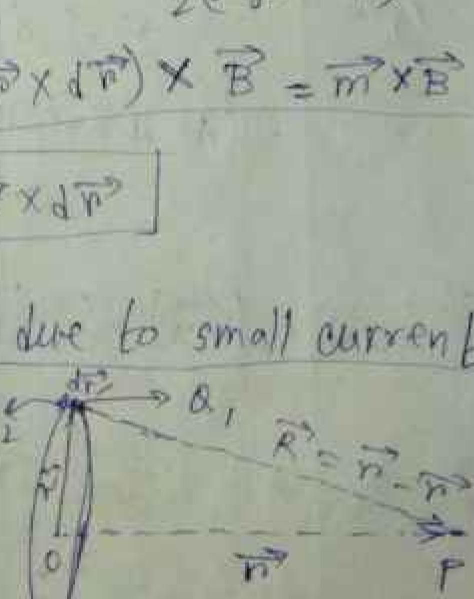
0 (Exact differential)

So, $\vec{\tau} = -\frac{I}{2}\vec{B}\times\oint\vec{r}\times d\vec{l} = \frac{1}{2}(I\oint\vec{r}\times d\vec{r})\times\vec{B}$

Hence, $\vec{\tau} = \left(\frac{I}{2}\oint\vec{r}\times d\vec{r}\right)\times\vec{B} = \vec{m}\times\vec{B}$

Here, $\boxed{\vec{m} = \frac{I}{2}\oint\vec{r}\times d\vec{r}}$

L $\vec{B}$ at large dist. due to small current loop:



$\vec{OQ_1} = \vec{r'}$

$\vec{OQ_2} = \vec{r'} + d\vec{r'}$

$d\vec{l} = d\vec{r'}$

$$\vec{A} = \frac{\mu_0}{4\pi}\oint\frac{I\,d\vec{l}}{R} \quad \text{--- (1)}$$

$$\vec{R} = \vec{r} - \vec{r'} \Rightarrow R = \sqrt{(\vec{r}-\vec{r'})\cdot(\vec{r}-\vec{r'})}$$

$$\Rightarrow R = (r^2 - 2\vec{r}\cdot\vec{r'} + r'^2)^{1/2}$$

$$\Rightarrow \frac{1}{R} = (r^2 - 2\vec{r}\cdot\vec{r'} + r'^2)^{-1/2}$$

$$= \cancel{r^2}\, r^{(-\frac{1}{2})\cdot 2}\left(1 - \frac{2\vec{r}\cdot\vec{r'}}{r^2} + \left(\frac{r'}{r}\right)^2\right)^{-1/2}$$

$$\Rightarrow \frac{1}{R} = \frac{1}{r}\left(1 + \frac{\vec{r}\cdot\vec{r'}}{r^2} + \cdots\right)$$

Higher order terms are neglected

(1) $\rightarrow$

$$\vec{A} = \frac{\mu_0 I}{4\pi}\left[\frac{1}{r}\oint d\vec{r'} + \frac{1}{r^3}\oint d\vec{r'}(\vec{r}\cdot\vec{r'}) + \cdots\right] \quad (2)$$

'0' for closed loop.

$$\vec{r}\times(d\vec{r'}\times\vec{r'}) = d\vec{r'}(\vec{r}\cdot\vec{r'}) - \vec{r'}(\vec{r}\cdot d\vec{r'}) \quad (3)$$

$$d[\vec{r'}(\vec{r}\cdot\vec{r'})] = d\vec{r'}(\vec{r}\cdot\vec{r'}) + \vec{r'}(\vec{r}\cdot d\vec{r'}) \quad (4)$$

(for change in $\vec{r'}$)

(3) + (4) $\rightarrow$ $d[\vec{r'}(\vec{r}\cdot\vec{r'})] + \vec{r}\times(d\vec{r'}\times\vec{r'})$

$$= 2\,d\vec{r'}(\vec{r}\cdot\vec{r'})$$

$$\rightarrow d\vec{r'}(\vec{r}\cdot\vec{r'}) = \frac{1}{2}\vec{r}\times(d\vec{r'}\times\vec{r'})$$

(4)

$$+\frac{1}{2}d[\vec{r'}(\vec{r}\cdot\vec{r'})]$$

(As it's exact diff, it gets 0 after ~~differ~~ integration)

(2) $\rightarrow$ $\cancel{\vec{A} = \left(\frac{\mu_0}{4\pi}\right)\frac{1}{r^3}\oint\frac{1}{2}d[\vec{r'}(\vec{r}\cdot\vec{r'})]}$

$$(2) \rightarrow \vec{A} = \frac{\mu_0 I}{4\pi}\frac{1}{r^3}\oint\frac{1}{2}\vec{r}\times(d\vec{r'}\times\vec{r'})$$

$$= \frac{\mu_0}{4\pi}\left[\frac{I}{2}\oint(\vec{r'}\times d\vec{r'})\right]\times\frac{\vec{r}}{r^3}$$

$(\vec{m})$

Hence, $\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\left(\frac{\vec{m}\times\vec{r}}{r^3}\right)}$

Area of the loop, $\underline{\vec{S} = \frac{1}{2}\oint(\vec{r}\times d\vec{r})}$

$\boxed{\vec{m} = I\vec{S}}$

Now, $\vec{B} = \vec{\nabla}\times\vec{A}$

$\Rightarrow \vec{B} = \frac{\mu_0}{4\pi}\vec{\nabla}\times\left(\frac{\vec{m}\times\vec{r}}{r^3}\right)$

$= \frac{\mu_0}{4\pi}\left[\frac{1}{r^3}\vec{\nabla}\times(\vec{m}\times\vec{r}) + \vec{\nabla}\frac{1}{r^3}\times(\vec{m}\times\vec{r})\right]$

Now, $[\vec{\nabla}\times(\vec{m}\times\vec{r})]_z = \frac{\partial}{\partial y}\cancel{(m_x y - m_y x)}$

$\cancel{\frac{\partial}{\partial z}(}$ $2m_z$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (\vec{m}\times\vec{r})_x & (\vec{m}\times\vec{r})_y & (\vec{m}\times\vec{r})_z \end{vmatrix}$$

$$[\vec{m}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ m_x & m_y & m_z \\ x & y & z \end{vmatrix} = \hat{i}(zm_y - ym_z) + \hat{j}(xm_z - zm_x) + \hat{k}(ym_x - xm_y)]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ zm_y - ym_z & xm_z - zm_x & ym_x - xm_y \end{vmatrix}$$

$\Rightarrow$ $\underline{z\text{ comp.}} = \frac{\partial}{\partial x}(xm_z - zm_x) - \frac{\partial}{\partial y}(zm_y - ym_z)$

$= m_z + m_z = 2m_z$

Similarly, x comp. $= 2m_x$

y comp. $= 2m_y$

So, $\vec{\nabla} \times (\vec{m} \times \vec{r}) = \hat{i}.2m_x + \hat{j}.2m_y + \hat{k}.2m_z$

$= 2\vec{m}$

$\cancel{\vec{\nabla}}$, $\vec{\nabla}\left(\frac{1}{r^3}\right) = \hat{r}\frac{\partial}{\partial r}\left(\frac{1}{r^3}\right) = -\frac{3\hat{r}}{r^4}$

$= \frac{-3\vec{r}}{r^5}$

$\vec{\nabla}\left(\frac{1}{r^3}\right) \times (\vec{m} \times \vec{r}) = -\frac{3}{r^5}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ zm_y - ym_z & xm_z - zm_x & ym_x - xm_y \end{vmatrix}$

$\boxed{\vec{m} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ m_x & m_y & m_z \\ x & y & z \end{vmatrix}}$

z comp. $= \cancel{-\frac{3}{r^4}}\ -\frac{3}{r^5}\left[x^2 m_z - zxm_x - yzm_y + y^2 m_z\right]$

$= -\frac{3}{r^5}\left[\cancel{(x^2+y^2)m_z}\ (x^2+y^2+z^2)m_z - z^2 m_z - zxm_x - yzm_y\right]$

$= -\frac{3}{r^5} m_z . r^2 + \frac{3}{r^5}(z^2 m_z + zxm_x + yzm_y)$

$= -\frac{3}{r^3} m_z + \frac{3}{r^5}(xm_x + ym_y + zm_z) z$

$= -\frac{3}{r^3} m_z + \frac{3}{r^5}(\vec{m}.\vec{r}) z$

So. $\vec{\nabla}\left(\frac{1}{r^3}\right) \times (m \times \vec{r}) = -\frac{3}{r^3}(m_x\hat{i} + m_y\hat{j} + m_z\hat{k}) + \frac{3}{r^5}(x\hat{i} + y\hat{j} + z\hat{k}).(\vec{m}.\vec{r})$

$= -\frac{3\vec{m}}{r^3} + \frac{3(\vec{m}.\vec{r})\vec{r}}{r^5}$

Hence, $\vec{B} = \frac{\mu_0}{4\pi}\left[\frac{2\vec{m}}{r^3} - \frac{3\vec{m}}{r^3} + \frac{3(\vec{m}\cdot\vec{r})\vec{r}}{r^5}\right]$

$$\Rightarrow \boxed{\vec{B} = \frac{\mu_0}{4\pi}\left[-\frac{\vec{m}}{r^3} + \frac{3(\vec{m}\cdot\vec{r})\vec{r}}{r^5}\right]}$$

1 Magnetic Scalar potential:

$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ If $\vec{J}=0 \Rightarrow \vec{\nabla}\times\vec{B}=0$

Here, we can write, $\vec{B} = -\mu_0\vec{\nabla}\phi$.

ϕ is magnetic scalar pot.

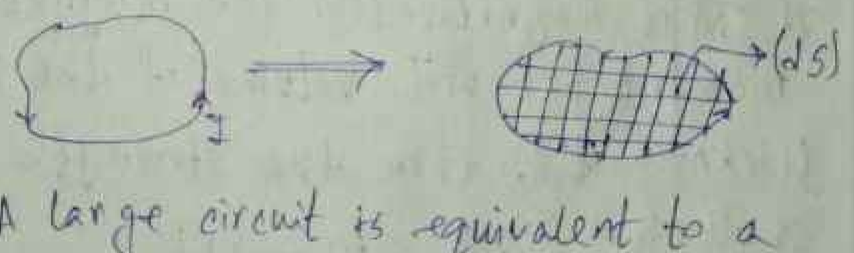
$$\vec{B} = \frac{\mu_0}{4\pi}\left(-\frac{\vec{m}}{r^3} + \frac{3(\vec{m}\cdot\vec{r})\vec{r}}{r^5}\right)$$

$$= -\frac{\mu_0}{4\pi}\vec{\nabla}\left(\frac{\vec{m}\cdot\vec{r}}{r^3}\right)$$

Hence, $\boxed{\phi = \frac{1}{4\pi}\frac{\vec{m}\cdot\vec{r}}{r^3}}$

Now, $\vec{\nabla}\cdot\vec{B} = 0 \rightarrow \boxed{\nabla^2\phi = 0}$

- Magnetic dipole moment, $\vec{m} = I\vec{A}$



A large circuit is equivalent to a large no. of elementary current loops.

$d\vec{m} = I\,d\vec{s} \leftarrow$ (infinitely small)

contribution to the scalar pot. at P due to an elementary loop,

$$d\phi = \frac{d\vec{m}\cdot\vec{r}}{4\pi r^3}$$

Total scalar pot. at P, $\phi = \int_s d\phi = \frac{1}{4\pi} \int \frac{d\vec{m} \cdot \vec{r}}{r^3}$

$\rightarrow \phi = \frac{I}{4\pi} \int \frac{d\vec{s} \cdot \vec{r}}{r^3}$

Solid angle subtended by elementary loop at P, $d\Omega = \frac{d\vec{s} \cdot \vec{r}}{r^3}$

So, $\phi = \frac{I}{4\pi} \int d\Omega \Rightarrow \boxed{\phi = \left(\frac{I}{4\pi}\right)\Omega}$

The whole circuit is equivalent to a large no. of magnetic dipoles or a sheet of magnetized materials ~~bounde~~ normally bounded by the ~~sheet~~ circuit.

Such a layer is known as magnetic double layer or magnetic shell.

__Magnetic shell or Ampere's equivalent theorem:__ The strength (τ) of the magnetic shell is defined as its magnetic dipole moments per unit facial area.

If $\vec{M}$ is magnetization (i.e. magnetic dipole moment per unit volume of the shell of thickness t), then the strength of the shell is, $\tau = Mt$

The strength of the equivalent magnetic shell would be, $\tau = \frac{\text{magnetic moment}}{\text{facial Area}}$

$= \frac{\text{current . area}}{\text{area}}$

$\Rightarrow \boxed{\tau = I}$

__Shell strength = current in loop__

Thus, ~~that~~ a current carrying loop produces same magnetic field as it's produced by a magnetic shell whose boundary ~~consi~~ co-inside with the boundary of the loop & whose ~~mag~~ strength is equal to the current flowing ~~to~~ in the loop.

This is Ampere's Equivalence Theorem.

Electromagnetic Induction.

(i) Induced emf $\propto$ rate of change ~~of m~~ in magnetic flux

(ii) ~~t~~ Dir.^{n} of induced emf is such that it tries to oppose the cause of its generation i.e. variation in magnetic flux. $\rightarrow$ (Lenz's Law)

ind. emf, $\boxed{\mathcal{E} = -\frac{d\phi_B}{dt}}$ — (1)

$$\phi_B = \int_S \vec{B}\cdot d\vec{S} \quad \text{— (3)}$$

$$\mathcal{E} = \oint_C \vec{E}\cdot d\vec{l} \quad \text{— (2)}$$

Hence, $\boxed{\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{S}}$ — (4)

Integral form of Faraday's Law.

$$\Rightarrow \int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{S} = -\int_S \frac{d\vec{B}}{dt}\cdot d\vec{S}$$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$ (Maxwell's 3rd Law)

Differential form of Faraday's Law.

Vec & Scalar Potential in time varying EM field:

For electrostatics as $\vec{\nabla}\times\vec{E} = 0$, we can write $\vec{E} = -\vec{\nabla}\phi$. But for presence of $\vec{B}(t)$, $\vec{\nabla}\times\vec{E} \neq 0$, so we can't write, $\vec{E} = -\vec{\nabla}\phi$.

Now, $\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A})$

$$\Rightarrow \vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$$

we can say, $\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\phi$

$$\Rightarrow \vec{E} = \underbrace{-\vec{\nabla}\phi}_{\vec{E_q}} \underbrace{- \frac{\partial \vec{A}}{\partial t}}_{\vec{E_i}}$$

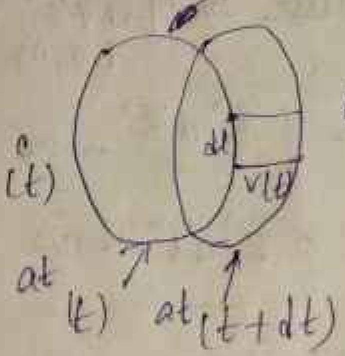
(field generated by charge) $\vec{E_q}$

$\vec{E_i}$ (field generated by $\vec{B}(t)$)

$\vec{E_q} = -\vec{\nabla}\phi$, $\vec{E_i} = -\frac{\partial \vec{A}}{\partial t}$ | $\vec{E} = \vec{E_q} + \vec{E_i}$

conservative & non solenoidal; non-conservative & solenoidal

Faraday's Law from moving loop:



e^- moves with $\vec{u}$ wrt wire.

vel. of e^- wrt $\vec{B}$, $(\vec{u} + \vec{v})$

force, $\vec{F} = q(\vec{u} + \vec{v}) \times \vec{B}$

$\frac{\text{force}}{\text{charge}}$ = motional ind. emf.

$\vec{E} = (\vec{u} + \vec{v}) \times \vec{B}$

$\mathcal{E} = \oint \vec{E} \cdot d\vec{l} = \oint (\vec{u} + \vec{v}) \times \vec{B} \cdot d\vec{l}$

$(\vec{u} \parallel d\vec{l}) \rightarrow \mathcal{E} = \oint (\vec{v} \times \vec{B}) \cdot d\vec{l}$

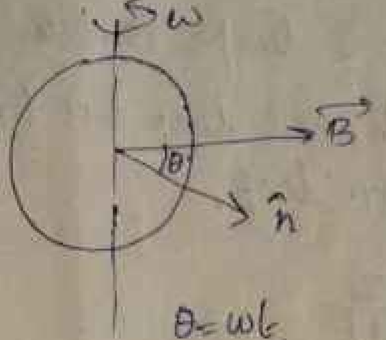
(Area swept by $d\vec{l}$ in time dt) = $\vec{v}\,dt \times d\vec{l}$

$d\phi_B = \vec{B} \cdot \vec{\text{Area}} \Rightarrow d\phi_B = \vec{B} \cdot (\vec{v}\,dt \times d\vec{l})$

$$\rightarrow \frac{d\phi_B}{dt} = -(\vec{v} \times \vec{B}) \cdot d\vec{l}$$

Hence, $\mathcal{E} = -\frac{d\phi}{dt}$

Emf in rotating loop in $\vec{B} = $ const. :



$\theta = \omega t$

$$\phi = \vec{B}\cdot\hat{n}\, S = BS\cos\theta$$
$$= BS\cos\omega t$$
$$\mathcal{E} = -\frac{d\phi}{dt} = -(-\sin\omega t)(BS\omega)$$
$$\rightarrow \boxed{\mathcal{E} = BS\omega\sin\omega t}$$

Induced emf in stationary loop & $\vec{B} \neq$ const:

For time varying $\vec{B}$, $\vec{B} = \vec{B}_0 \sin\omega t$

$$\mathcal{E} = \oint \vec{E}\cdot d\vec{l} = \int(\vec{\nabla}\times\vec{E})\cdot d\vec{S}$$
$$= -\int \frac{\partial B}{\partial t}\, dS = -\int B_0\omega\cos\omega t\, dS$$
$$\rightarrow \underline{\mathcal{E} = -B_0 S\omega\cos\omega t}$$

this emf is often called transformer emf.

Self Inductance: for ckt in non-feromagnetic medium, $\phi \propto I$

$\therefore \phi = LI$, $L =$ co-eff of self inductance

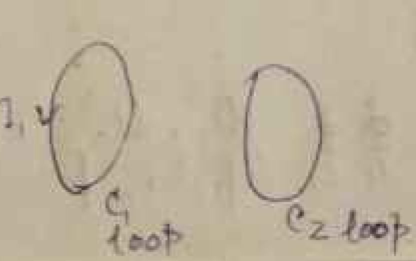
$$\mathcal{E} = -\frac{d\phi}{dt} \rightarrow \mathcal{E} = -L\frac{dI}{dt}$$

& $$\mathcal{E} = -\frac{d\phi}{dI}\cdot\frac{dI}{dt} = -L\frac{dI}{dt}$$

Here, $L = \frac{d\phi}{dI}$ is called incremental self-inductance.

SI unit of L is H. (Henry).

I Mutual Inductance:



$\phi_2 \propto I_1 \Rightarrow \phi_2 = M_{21} I_1$ & $\phi_1 = M_{12} I_2$

Mutual Inductance

So, Mutual inductance of 2 loops is ~~the~~ equal to flux linked with one loop due to unit current in the other loop.

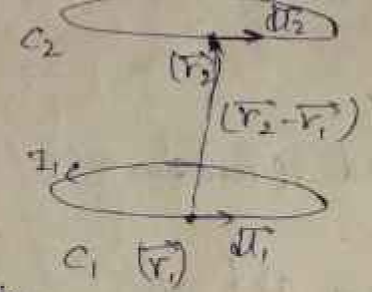
emf induced in one due to other

$$\mathcal{E}_2 = -\frac{d\phi_2}{dt} = -M_{21}\frac{dI_1}{dt}$$

$$\mathcal{E}_1 = -\frac{d\phi_1}{dt} = -M_{12}\frac{dI_2}{dt}$$

Unit of mutual inductance $\rightarrow$ (SI) H (Henry)

Neumann's formula:



Origin

$$d\phi_2 = \vec{B_1}\cdot\vec{dS_2}$$

$$\phi_2 = \int \vec{B_1}\cdot\vec{dS_2} = \int(\vec{\nabla}\times\vec{A_1})\cdot\vec{dS_2}$$

$$\Rightarrow \phi_2 = \oint_{C_2} \vec{A_1}\cdot\vec{dl_2}$$

Now vec. pot., $\vec{A_1} = \frac{\mu_0}{4\pi}\oint_{C_1}\frac{I_1\,\vec{dl_1}}{|\vec{r_2}-\vec{r_1}|}$

$$\phi_2 = \oint_{C_2}\oint_{C_1}\frac{\mu_0}{4\pi}\frac{I_1\,\vec{dl_1}}{|\vec{r_2}-\vec{r_1}|}\cdot\vec{dl_2}$$

$$\rightarrow \phi_2 = \left(\frac{\mu_0}{4\pi}\oint_{C_2}\oint_{C_1}\frac{\vec{dl_1}\cdot\vec{dl_2}}{|\vec{r_2}-\vec{r_1}|}\right)I_1 \equiv \phi_2 = M_{21}I_1$$

So, $$M_{21} = \frac{\mu_0}{4\pi}\oint_{C_2}\oint_{C_1}\frac{\vec{dl_1}\cdot\vec{dl_2}}{|\vec{r_2}-\vec{r_1}|}$$

Similarly, we have, $\phi_1 = \oint_{C_1}\vec{A_2}\cdot\vec{dl_1}$, $\vec{A_2} = \frac{\mu_0}{4\pi}\oint$

$$\phi_1 = \frac{\mu_0}{4\pi}$$

$$M_{12} = \frac{\mu_0}{4\pi}\oint_{C_1}\oint_{C_2}\frac{\vec{dl_1}\cdot\vec{dl_2}}{|\vec{r_1}-\vec{r_2}|}$$

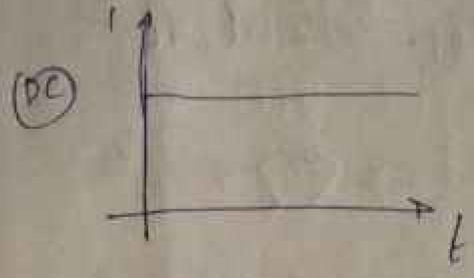
So, M_{21} & M_{12} are purely geometrical quantity.
It depends on shape of loops, rel. position of loops.

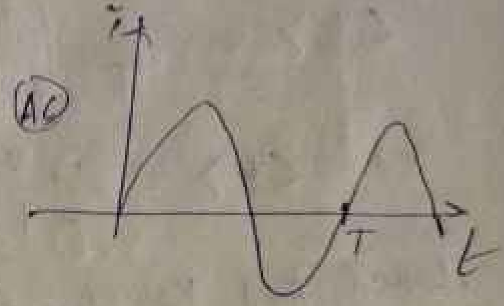
$$\boxed{M_{12} = M_{21} = M}$$

Mutual inductance b/w C_1 & C_2.

So, $$M = \frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_2} \frac{\vec{dl_1} \cdot \vec{dl_2}}{|\vec{r_1} - \vec{r_2}|}$$

Alternating Current





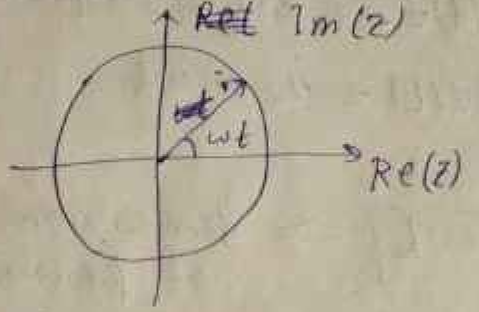
Lin. freq. $\nu = \frac{1}{T}$, Ang. freq, $\omega = \frac{2\pi}{T}$

$i(t)$ should be periodic.

$i_1(t) = i_0 \sin\omega t$, $i_2(t) = i_0 \sin(\omega t + \alpha)$ ← phase angle

($i_2(t)$ leads $i_1(t)$ by α)

$i(t) = i_0 e^{j\omega t}$ → $i(t) = \cos\omega t + j \sin\omega t$ $(j = \sqrt{-1})$



~~Re~~

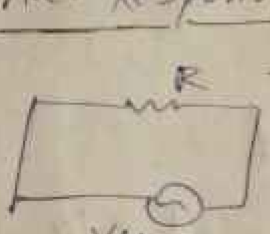
$Re(z) = i_0 \cos\omega t = Re(i)$

$Im(z) = i_0 \sin\omega t = Im(i)$

($i(t) = e^{j(\omega t + \phi)}$ leads $i(t) = e^{j\omega t}$ by an angle ϕ.)

A.C. Response in C, R, L:

Resistance:



$V(t) = V_0 \sin\omega t$

Ohm's Law: $V_0 \sin\omega t = i(t) R$

$\Rightarrow i(t) = \frac{V_0}{R} \sin\omega t$

$i_0 = \frac{V_0}{R}$ $\Rightarrow i(t) = i_0 \sin\omega t$

$V(t)$ & $i_0(t)$ are in same phase.

Power, $P(t) = i(t) V(t) \Rightarrow P(t) = i_0 V_0 \sin^2\omega t$

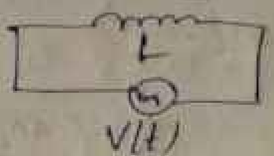
$\Rightarrow P(t) = \frac{i_0 V_0}{2}(1 - \cos 2\omega t)$

$$\langle P \rangle = \frac{1}{T}\int_0^T \frac{i_0 V_0}{2}(1 - \cos 2\omega t)\,dt$$

$$\Rightarrow \langle P \rangle = \frac{i_0 V_0}{2T} \cdot T \Rightarrow \langle P \rangle = \frac{i_0 V_0}{2}$$

Hence, Avg. Power, $\underbrace{\langle P \rangle = \frac{i_0 V_0}{2} = \frac{i_0^2 R}{2}}_{\text{Joule heat}}$

Inductor:



$V_L = -L\frac{di}{dt}$

$V(t) - L\frac{di}{dt} = 0$ | Taking internal Rest of Inductor $= 0$

$\Rightarrow V(t) = L\frac{di}{dt}$

If $i = i_0 \sin(\omega t)$, | $V(t) = i_0 \omega L \cos \omega t$

$\Rightarrow V(t) = V_0 \cos \omega t$

$V_0 = i_0 \omega L$

$(V = iR)$

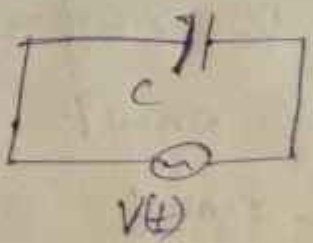
$\boxed{\text{Reactance} = \omega L} \rightarrow$ dimension of Resistance

Hence, $V(t) = V_0 \sin(\omega t + \frac{\pi}{2})$

$i(t) = i_0 \sin \omega t$

So, here, $V(t)$ leads $i(t)$ by $\frac{\pi}{2}$

Capacitor:



$V(t) = V_0 \sin \omega t$

$i = \frac{dq}{dt} \Rightarrow i\,dt = dq$

$\Rightarrow i\,dt = C\,dV$

$\Rightarrow i = C\frac{dV}{dt} = C\omega V_0 \cos \omega t$

$\Rightarrow i = i_0 \cos \omega t$

So, $i_0 = \omega C V_0$

$\boxed{\text{Reactance} = \frac{1}{\omega C}}$

dimension of Resistance

$i(t) = i_0 \sin(\omega t + \frac{\pi}{2})$, $V(t) = V_0 \sin \omega t$.

~~...~~ $i(t)$ leads ~~by~~ $V(t)$ by $\cdot \frac{\pi}{2}$

Effective/ rms value:

for $f(t)$, effective, $f_{eff} = \dfrac{\int_0^{T/2} f(t)dt}{\int_0^{T/2} dt}$

$$f_{rms} = \left[\frac{\int_0^T f^2\, dt}{\int_0^T dt}\right]^{1/2}$$

for $V = V_0 \sin \omega t$,

$$V_{rms}^2 = \frac{V_0}{T}\int_0^T \sin^2 \frac{2\pi}{T}t\, dt = \frac{V_0}{T}\int_0^T \frac{\left(1 - \cos\frac{4\pi t}{T}\right)dt}{2}$$

$$\rightarrow V_{rms}^2 = \frac{V_0}{T}\cdot\frac{T}{2} \Rightarrow \boxed{V_{rms} = \frac{V_0}{\sqrt{2}}}$$

for $i = i_0 \sin \omega t$, $\boxed{i_{rms} = \frac{i_0}{\sqrt{2}}}$

Now, $V_0 = \sqrt{2}\, V_{rms} = \sqrt{2} \times 220$

so, $V(t) = \sqrt{2}\cdot 220 \sin \omega t$ $\quad$ [$f = 50\,Hz$

$\rightarrow V(t) = \sqrt{2} \times 220 \sin 100\pi t$ $\quad$ $\omega = 100\pi \frac{rad}{s}$]

for V_{eff}, $V_{eff} = \dfrac{V_0 \int_0^{T/2} \sin \omega t\, dt}{\int_0^{T/2} dt}$

$$\Rightarrow V_{eff} = \frac{V_0}{T/2} \cdot \frac{1}{\omega}\left[-\cos \omega t\right]_0^{T/2}$$

$$\rightarrow V_{eff} = \frac{2V_0}{\omega T}\left[\cos 0 - \cos \frac{2\pi}{T}\cdot\frac{T}{2}\right]$$

$$\rightarrow V_{eff} = \frac{2V_0}{\omega T}(1+1) = \frac{4V_0}{\omega T} = \frac{4V_0}{2\pi \omega}\omega$$

$\rightarrow \boxed{V_{eff} = \frac{2V_0}{\pi}}$ Similarly, $\boxed{i_{eff} = \frac{2i_0}{\pi}}$

Series LCR ckt with ac source:

v(t)

$v(t) = V_0 \sin \omega t$

$\equiv v(t) = V_0 e^{j\omega t}$

$$V(t) = V_L + V_C + V_R = L\frac{di}{dt} + \frac{q}{C} + iR$$

$$\Rightarrow \frac{dV}{dt} = L\frac{d^2i}{dt^2} + \frac{1}{C}\frac{dq}{dt} + \frac{di}{dt}R$$

$$\Rightarrow V_0 j\omega e^{j\omega t} = L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C}$$

Let $i = A e^{j\omega t}$ is trial sol^n

So, $\frac{di}{dt} = Aj\omega e^{j\omega t}$, $\frac{d^2i}{dt^2} = -A\omega^2 e^{j\omega t}$

So, $V_0 j\omega e^{j\omega t} = -L.A\omega^2 e^{j\omega t} + R.Aj\omega e^{j\omega t} + \frac{A}{C}e^{j\omega t}$

$$\rightarrow A\left(-\omega^2 L + j\omega R + \frac{1}{C}\right) = j\omega V_0$$

$$\Rightarrow A = \frac{j\omega V_0}{-\omega^2 L + j\omega R + \frac{1}{C}}$$

$$\Rightarrow A = \frac{V_0}{-\frac{\omega L}{j} + R + \frac{1}{j\omega C}}$$

$$\Rightarrow A = \frac{V_0}{j\omega L + R - \frac{j}{\omega C}}$$

$$\Rightarrow A = \frac{V_0}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

Hence, $$i(t) = \frac{V_0 e^{j\omega t}}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

Here, ~~Reactance~~, $Z = R + j\left(\omega L - \frac{1}{\omega C}\right)$

Impedence

$Z = |Z| e^{j\theta}$ phase angle.

$|Z| = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}$, $\tan\theta = \frac{\omega L - \frac{1}{\omega C}}{R}$

$$i(t) = \frac{V_0 \quad e^{j\omega t}}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}\, e^{j\theta}}$$

$$\Rightarrow \quad i(t) = \frac{V_0\, e^{j(\omega t - \theta)}}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}} = i_0\, e^{j(\omega t - \theta)}$$

Here, $i_0 = \frac{V_0}{|Z|}$

for, $Z = R + j(\omega L - \frac{1}{\omega C})$

(i) $\omega L > \frac{1}{\omega C} \Rightarrow \tan\theta > 0 \Rightarrow \theta > 0$

(inductive reac) (capacitive reac.) so, $i(t) = i_0 e^{j(\omega t - \theta)}$

$V(t) = V_0 e^{j\omega t}$

V(t) leads i(t) by θ.

(ii) $\omega L < \frac{1}{\omega C} \Rightarrow \tan\theta < 0 \Rightarrow \theta < 0$

so, $i(t) = i_0 e^{j(\omega t + \theta)}$, $V(t) = V_0 e^{j\omega t}$

so, i(t) leads V(t) by θ.

(iii) $\omega L = \frac{1}{\omega C} \Rightarrow \tan\theta = 0 \Rightarrow \theta = 0$

so, $i(t) = i_0 e^{j\omega t}$, $V(t) = V_0 e^{j\omega t}$

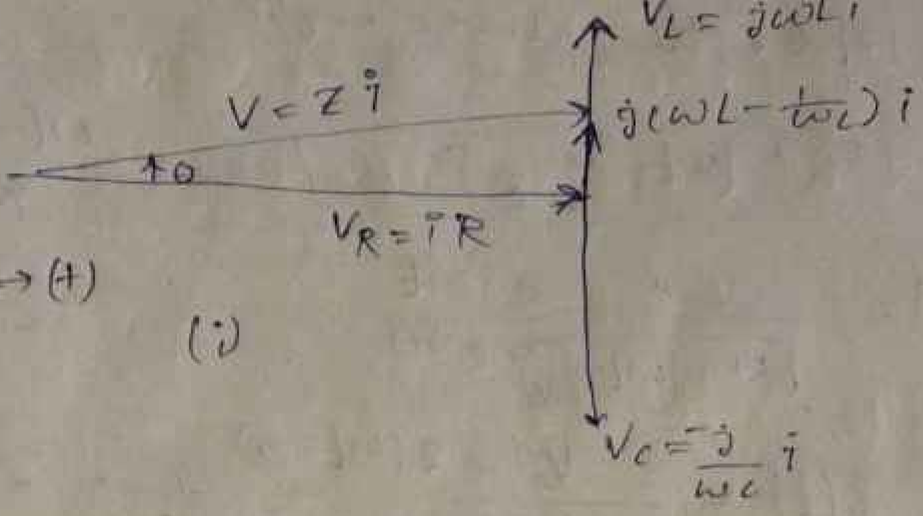
so, i(t) & V(t) ~~has~~ have same phase.

Here, $|Z| = R \longrightarrow$ ckt is completely resistive.

$i_0 = V_0 / R$

Here, $Z = \sqrt{R^2 + (\omega L - 1/\omega C)^2}$ is min.

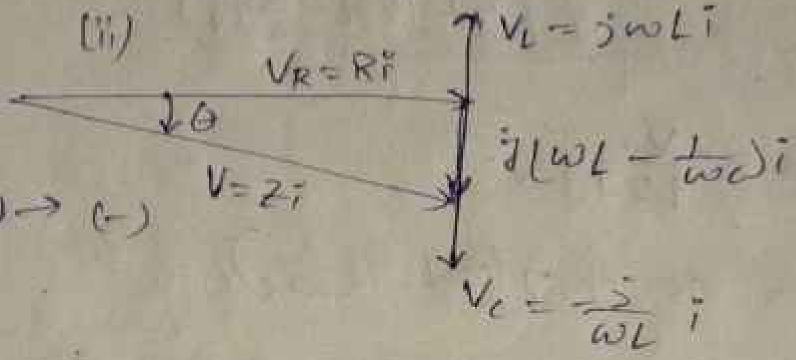
$|Z| = \min. \Rightarrow i = \max$ This is called resonance.



$\theta \rightarrow (+)$

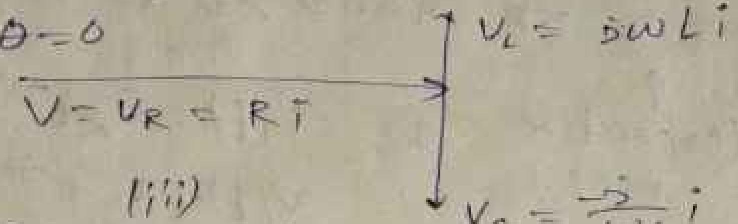
(i)

(ii)



$\theta \rightarrow (-)$

$\theta = 0$



(iii)
(Resonance)

$$\text{Aditance} = \frac{1}{\text{Impedence}}$$

$$\rightarrow \quad Y = \frac{1}{Z}.$$

<u>Series resonance</u>: Let at $\omega = \omega_0$,

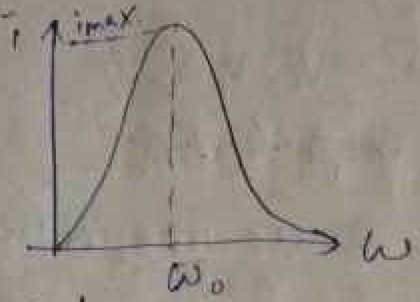
$$Im(Z) = 0 \Rightarrow Z = R + i(\omega_0 L - \frac{1}{\omega_0 C}) = R$$

So, $\omega_0 L - \frac{1}{\omega_0 C} = 0 \Rightarrow \boxed{\omega_0 = \frac{1}{\sqrt{LC}}}$

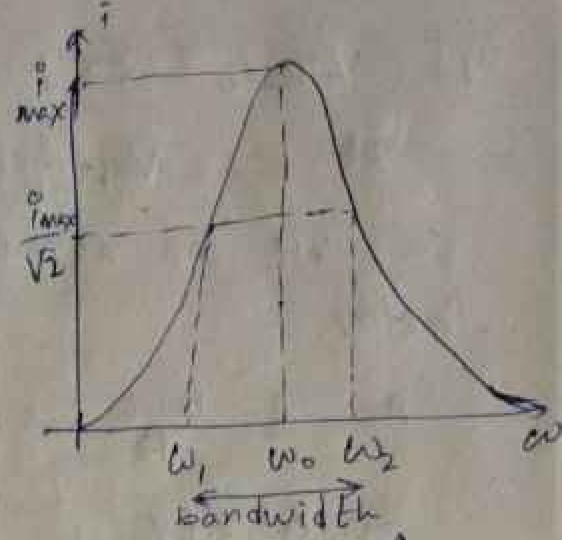
So, $2\pi \nu_0 = \frac{1}{\sqrt{LC}} \Rightarrow \boxed{\nu_0 = \frac{1}{2\pi\sqrt{LC}}}$

Resonance frequency

Here, the ckt is called acceptor ckt.

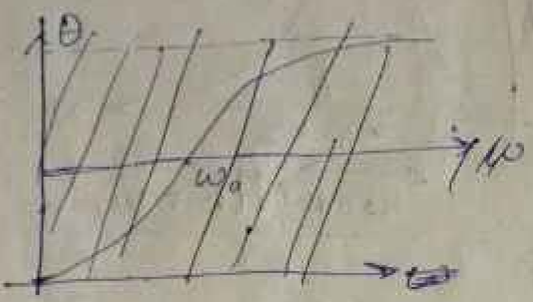


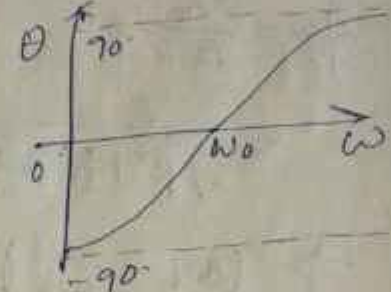
Current response curve of a series LCR ckt.



Sharpness of resonance $\downarrow$ $\Rightarrow$ $R \uparrow$

Sharpness $\uparrow$ is needed & more useful.





L

Now,

voltage drop accross R, $V_R = i_{max} R$

" " " L, $V_L = j\omega_0 L \cdot i_{max}$

" " " C, $V_C = \frac{-j}{\omega_0 C} i_{max}$

so, $V_L = j\omega_0 L \cdot \frac{V}{R}$, $V_C = \frac{-j}{\omega_0 C} \cdot \frac{V}{R}$

$V_C + V_L = 0$ at resonance

Phase diff b/w V_C & V_L = 180°

Quality factor of series LCR:

Q-factor, $\boxed{Q = \frac{\omega_0 L}{R}}$ $\left(\frac{\text{Inductive Reactance}}{\text{resistance}}\right)$ at resonance

Now, $\omega_0 = \frac{1}{\cancel{2\pi}\sqrt{LC}}$

$Q = \frac{1}{\sqrt{LC}} \cdot \frac{L}{R} \Rightarrow \underline{Q = \frac{1}{R}\sqrt{\frac{L}{C}}}$

$\omega_0 L = \frac{1}{\omega_0 C} \Rightarrow \boxed{|V_L| = |V_C| = QV_R}$

If $\cancel{V_L = V_C}$ $Q > 1 \Rightarrow V_L, V_C > V$

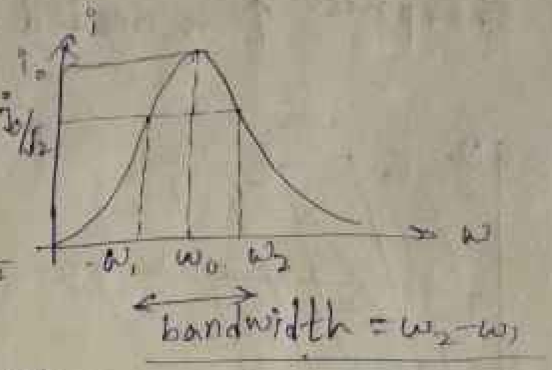
$\cancel{\#}$ Voltage magnification

So, we can name Q as magnification factor.

As $Q = \frac{\omega_0 L}{R} \Rightarrow$ when $R\uparrow$, $Q\downarrow$

Selectivity of a freq. $\downarrow$ $\Longleftarrow$ sharpness of resonance $\downarrow$

Bandwidth:



$$\frac{i_0}{\sqrt{2}} = \frac{V}{\sqrt{2}R} \Rightarrow$$

$$= \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

$\Rightarrow \left(\omega L - \frac{1}{\omega C}\right)^2 = R^2$

$\Rightarrow \omega L - \frac{1}{\omega C} = \pm R$

$\Rightarrow \omega L = \frac{1}{\omega C} + R, \frac{1}{\omega C} - R$

$\omega^2 L - \omega R - \frac{1}{C} = 0$

$$\omega = \frac{R \pm \sqrt{R^2 + \frac{4L}{C}}}{2L}$$

$\omega^2 L + \omega R - \frac{1}{C} = 0$

$$\omega = \frac{-R \pm \sqrt{R^2 + \frac{4L}{C}}}{2L}$$

~~Not acceptable~~

So, $$\omega_2 = \frac{R + \sqrt{R^2 + \frac{4L}{C}}}{2L}$$

& $$\omega_1 = \frac{-R + \sqrt{R^2 + \frac{4L}{C}}}{2L}$$

As, $\omega > 0$ always

Hence, $\omega_2 - \omega_1 = \cancel{\frac{2\sqrt{R^2 + 4L/C}}{2L}}$ $\frac{2R}{2L}$

$\rightarrow \omega_2 - \omega_1 = \frac{R}{L} = \frac{\omega_0 R}{\omega_0 L} = \frac{\omega_0}{Q}$

$\Rightarrow \boxed{Q = \frac{\omega_0}{\Delta\omega}}$ Hence, $\Delta\omega\downarrow \Rightarrow Q\uparrow$

Again, $Q = 2\pi \cdot \dfrac{\text{max. en. stored per cycle}}{\text{en. dissipated per cycle}}$

for, $V = V_0 \cos\omega t$, $i = i_0 \cos(\omega t - \theta)$

$p = Vi \Rightarrow p = V_0 i_0 \cos\omega t \cos(\omega t - \theta)$

$\Rightarrow p = V_0 i_0 (\cos^2\omega t \cos\theta + \cos\omega t \sin\omega t \sin\theta)$

$\langle P \rangle = \frac{1}{T}\int_0^T P\,dt = \frac{V_0 i_0 \cos\theta}{T}\underbrace{\int_0^T \cos^2\omega t\,dt}_{\frac{1}{2}} + \frac{V_0 i_0 \sin\theta}{T}\underbrace{\int_0^T \sin\omega t \cos\omega t\,dt}_{0}$

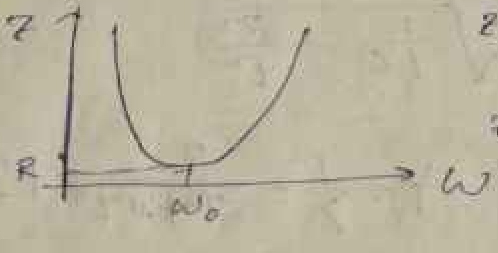
$\Rightarrow \langle p \rangle = \frac{V_0 i_0}{T} \cdot \frac{T}{2}\cos\theta = \frac{V_0}{\sqrt{2}} \frac{i_0}{\sqrt{2}} \cos\theta$

$\Rightarrow \underline{\langle p \rangle = V_{rms} i_{rms} \cos\theta}$

$\rightarrow \underline{P_{real} = P_{apparent} (\cos\theta)}$

for $\cos\theta = 1$, $\underline{P_{real} = P_{apparent}}$

Wattless power.

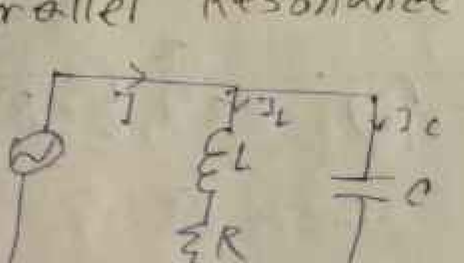


$z = R + j\left(\omega L - \frac{1}{\omega C}\right)$

$\cancel{z = \sqrt{R^2 + \left(\frac{V}{\omega L}\right)} / \frac{1}{}}$

$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$

Parallel Resonance LCR ckt:



$I = I_L + I_C$

$\Rightarrow \frac{V}{Z} = I_L + I_C$

$\Rightarrow \frac{V}{Z} = \frac{V}{R + j\omega L} + \frac{V}{\left(\frac{1}{j\omega C}\right)}$

$\Rightarrow \frac{V}{Z} = \frac{V}{R + j\omega L} + j\omega C V$

Taking $\underline{V = V_0 e^{j\omega t}}$;

$$\frac{1}{Z} = \frac{1}{R + j\omega L} + j\omega C$$

$$\rightarrow \frac{1}{Z} = \frac{R - j\omega L}{R^2 + \omega^2 L^2} + j\omega C$$

$$\rightarrow \cancel{\frac{1}{Z} = \frac{R}{R^2 + \omega^2 L^2} + j \frac{-\omega L + \omega^2 C L}{}} \quad \cancel{\frac{\omega C - \omega L}{R^2 + \omega^2 L^2}}$$

$$\rightarrow \frac{1}{Z} = \frac{R}{R^2 + \omega^2 L^2} + j\frac{\omega C R^2 + \omega^3 L^2 C - \omega L}{\omega^2 L^2 + R^2}$$

At resonance $\quad Im\left(\frac{1}{Z}\right) = 0$

$$\rightarrow \omega_p C R^2 + \omega_p^3 C L^2 - \omega_p L = 0$$

$$\rightarrow \omega_p^2 C L^2 = L - C R^2$$

$$\rightarrow \omega_p^2 = \frac{1}{LC} - \frac{R^2}{L^2}$$

$$\rightarrow \underline{\omega_p = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}}$$

Parallel resonance frequency

$$\nu_p = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

Now admittance, $\underline{Y = \frac{1}{Z} = \frac{R}{R^2 + \omega^2 L^2} + j\frac{\omega C R^2 + \omega^3 L^2 C - \omega L}{R^2 + \omega^2 L^2}}$

at resonance,

$$\underline{Y_p = \frac{R}{R^2 + \omega^2 L^2}}$$

$$\rightarrow Y_p = \frac{R}{R^2 + L^2\left(\frac{1}{LC} - \frac{R^2}{L^2}\right)} = \frac{R}{R^2 + \frac{L}{C} - R^2}$$

$$\rightarrow \boxed{Y_p = \frac{CR}{L}}$$

At resonance, $\frac{1}{Z_P} = Y_P \Rightarrow \underline{Z_P = \frac{L}{CR}}$

Impedence at resonance

for small R, $Z_P \uparrow$

for $R \to 0 \Rightarrow \omega_P \to \frac{1}{\sqrt{LC}} \to$ series LCR ckt

Line current at parallel resonance,

$$I_P = \frac{V}{Z_P}$$

Capacitor current at parallel resonance,

$$I_{CP} = \frac{V}{1/\omega_P C} = \omega_P C V$$

$$\frac{I_{CP}}{I_P} = \frac{\omega_P C}{1/Z_P} \Rightarrow \frac{I_{CP}}{I_P} = \omega_P C . Z_P$$

$$= \omega_P C \frac{L}{CR}$$

$$\Rightarrow \boxed{\frac{I_{CP}}{I_P} = \omega_P . \frac{L}{R} = Q}$$ i.e. Quality factor

Now,

$$Q = 2\pi \frac{\text{max. En. stored per cycle}}{\text{En. dissipated per cycle}}$$

$$\Rightarrow Q = 2\pi . \frac{\frac{1}{2} CV^2 \times 2L\omega_P}{\cancel{2\pi} CV^2 R} = 2\pi \frac{CV^2 L\omega_P}{CV^2 R}$$

$$\Rightarrow Q = \frac{\omega_P L}{R}$$

Now, $Y_P = \frac{1}{Z_P} = \frac{R}{R^2 + \omega_P^2 L^2}$

$$\Rightarrow Z_P = \frac{R^2 + \omega_P^2 L^2}{R} = R\left(1 + \frac{\omega_P^2 L^2}{R^2}\right)$$

$$\simeq R \frac{\omega_P^2 L^2}{R^2}$$

$$\Rightarrow Z_P = Q . \omega_P L$$

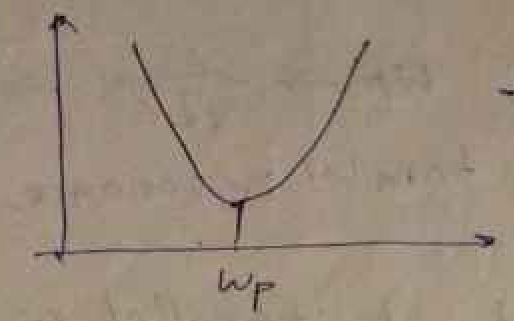
(for small R)

For $\omega \neq \omega_P$,

$$|Y|^2 = \frac{R^2}{(R^2+\omega^2 L^2)^2} + \frac{(\omega C R^2 + \omega^3 L^2 C - \omega L)^2}{(R^2+\omega^2 L^2)^2}$$

$$\frac{d|Y|^2}{d\omega} = 0 \Rightarrow \frac{d}{d\omega}\left[\left(\frac{1}{(R^2+\omega^2 L^2)^2}\right) + \left(\frac{\omega C R^2 + \omega^3 L^2 C - \omega L)^2}{R^2+\omega^2 L^2}\right)\right] = 0$$

$$\rightarrow \omega_m = \sqrt{\frac{1}{LC}\left(\frac{2CR^2}{L}+1\right)^{1/2} - \frac{R^2}{L^2}}$$



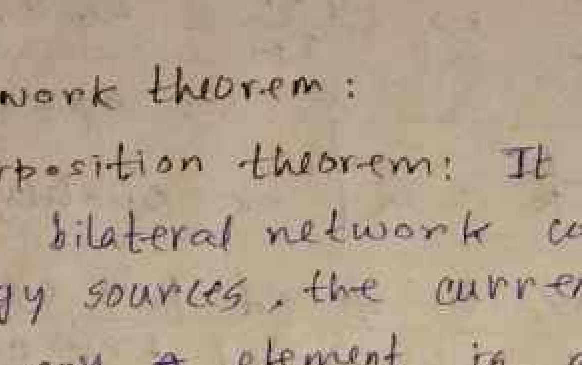
→ Resector ckt.

For band-width = $\Delta\omega = \omega_2 - \omega_1$,

$$Q = \frac{\omega_p}{\Delta\omega}$$

Network theorem :

1. superposition theorem: It states that in linear bilateral network containing several energy sources, the current/voltage in ~~gain~~ any element is algebric sum of the effect produced by each source acting separately. While considering the effect of a particular source, other sources must be replaced by their internal resistance.



(a)

1. KVL → $I_1R_1 + I_1r_1 + I_3R_3 = V_1$ — (1)

2. KVL → $-R_2I_2 - r_2I_2 - I_3R_3 = -V_2$

$\rightarrow (r_2+R_2)(I_3-I_1) + I_3R_3 = V_2$

$\Rightarrow -(r_2+R_2)I_1 + (r_1+R_2+R_3)I_3 = V_2$ — (2)

(1) → $(r_1+R_1)I_1 + R_3 I_3 - V_1 = 0$

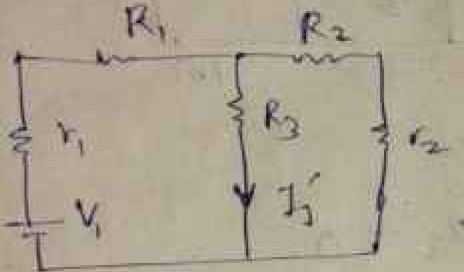
(2) → $-(r_2+R_2)I_1 + (r_1+R_2+R_3)I_3 - V_2 = 0$

$$I_1 = \frac{-R_3V_2 + V_1r_1 + V_1R_2 + V_1R_3}{r_1^2 + r_1R_2 + r_1R_3 + R_1r_1 + R_1R_2 + R_1R_3 + r_2R_3 + R_2R_3}$$

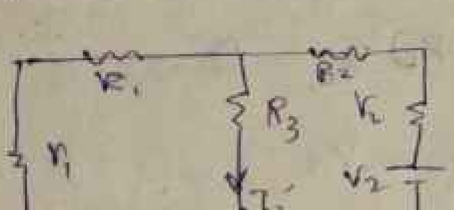
$$\rightarrow I_1 = \frac{-R_3V_2 + V_1(r_1+R_2+R_3)}{\Delta} \quad \Delta = \left| \quad \right|$$

We can write,

$$I_1 = \frac{1}{\Delta}\left| \quad \right|, \quad I_3 = \frac{1}{\Delta}\left| \quad \right|,$$



$$I_1' = \frac{1}{\Delta}\begin{vmatrix} (r_1+R_1) & V_1 \\ -(r_2+R_2) & 0 \end{vmatrix}$$



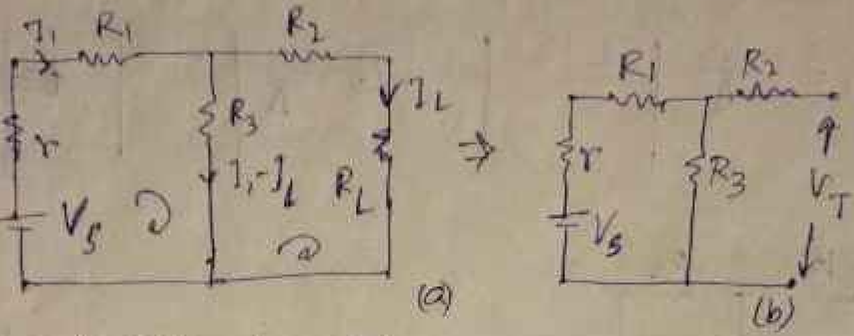
$$I_2' = \frac{1}{\Delta}\begin{vmatrix} (r_1+R_1) & 0 \\ -(r_2+R_2) & V_2 \end{vmatrix}$$

$\underline{I_3 = I_1' + I_2'}$ (verified)

↓

superposition principle

Thevnin Theorem: Any 2 terminal linear network containing energy sources & resitance can be replaced by a ~~simple~~ single equivalent ~~source~~ circuit containing a voltage source V_t together with a series resistance R_t, where V_t is the open ckt voltage b/w the terminals of the network & R_t is the resistance measured b/w the terminals with all energy source replaced by their internal resistance. The term volt-source



$$I_1(r+R_1) + (I_1 - I_L)R_3 = V_s$$

$$(R_2+R_L)\cancel{R_2}\, I_L - (I_1 - I_L)R_2 = 0$$

$$\Rightarrow -I_1 R_3 + (R_2+R_3+R_L) I_L = 0$$

$$I_1(r+R_1+R_3) - R_3 I_L = V_s$$

$$\cancel{I_L = \frac{1}{\Delta}\begin{vmatrix} -R_3 & 0 \\ r+R_1+R_3 & V_s \end{vmatrix}}, \quad \Delta = \cancel{\begin{vmatrix} -R_3 & R_2+R_3+R_L \\ r+R_1+R_3 & \end{vmatrix}}$$

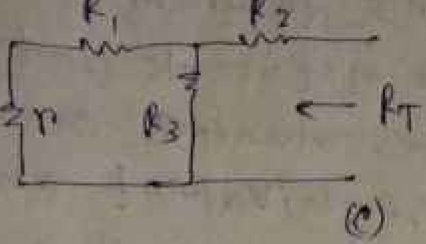
$$\Rightarrow \cancel{\left(\frac{R_2+R_3+R_L}{R_3}\right) I_L (R_1+R_3+r) - R_3 I_L = V_s}$$

$$\Rightarrow \cancel{\frac{I_L}{R_3}} \Rightarrow \frac{(R_2+R_3+R_L)(r+R_1+R_3)}{R_3} I_L - R_3 I_L = V_s$$

$$\Rightarrow I_L\left[(R_2+R_L)(r+R_1+R_3) + R_3(r+R_1) + R_3^2 - R_3^2\right] = V_s R_3$$

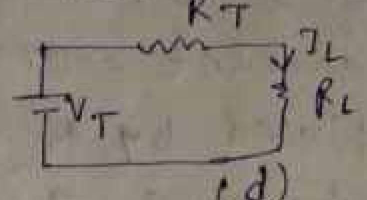
$$\Rightarrow I_L\left[(r+R_1+R_3)\left(R_2+R_L+\frac{R_3(r+R_1)}{r+R_1+R_3}\right)\right] = V_s R_3$$

$$\Rightarrow \boxed{I_L = \frac{V_s R_3}{(r+R_1+R_3)\left(R_2+R_L+\frac{R_3(r+R_1)}{r+R_1+R_3}\right)}}$$



(c)

Equiv ckt:



(d)

~~open ckt volt~~

(b) $\longrightarrow \quad V_T = \dfrac{V_S \cdot R_3}{r+R_1+R_3}$

(c) $\longrightarrow R_T = R_2 + \dfrac{R_3(r+R_1)}{r+R_1+R_3}$

(d) $\longrightarrow \quad (R_T+R_L)\cdot I_L = V_T$

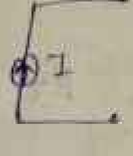
$\Rightarrow \quad I_L = \dfrac{V_T}{R_L + \left(R_2 + \dfrac{R_3(r+R_1)}{r+R_1+R_3}\right)}$

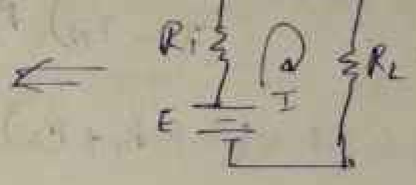
$\Rightarrow \quad I_L = \dfrac{V_S R_3}{(r+R_1+R_3)\left[R_L + R_2 + \dfrac{R_3(r+R_1)}{r+R_1+R_3}\right]}$

2. Current source:

(Ideal)

$I = \dfrac{E}{R_i+R_L}$





Voltage at the terminal, $V_L = IR_L = \dfrac{ER_i}{R_i+R_L}$

If Internal rest. $R_i >> R_L$, $V_L = \dfrac{ER_L}{R_i}$

$I = \dfrac{E}{R_i}$

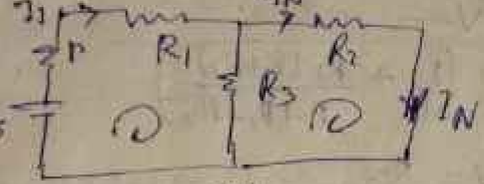
so, voltage source act as current source for $R_i >> R_L$.

[so, Internal rest. of an ideal current source $= \infty$, whereas that of an ideal volt. source $= 0$.]

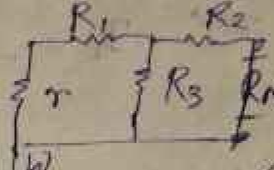
$I = \dfrac{E}{R_i}$ = const. for current source.

$V_L = \dfrac{ER_L}{R_i}$ changes by the change of R_L.

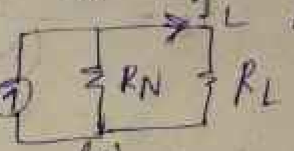
Norton's theorem: Acc. to this theorem, any 2 terminal linear network containing ~~a~~ current sources & resistances can be replaced by a simple equivalent ckt. containing a current source (I_N) in parrallel with a resistance R_N where I_N is the short ckt. current b/w the terminals of the network & R_N is a resistance measured b/w the terminals with all the ~~sour~~ energy sources replaced by their internal resistances.



(a)



(b)



(c)

KVL ⇓

$I_1 r + I_1 R_1 + (I_1 - I_N) R_3 = V_s$

$\Rightarrow (r + R_1 + R_3) I_1 - R_3 I_N = V_s$

& $I_N R_2 - (I_1 - I_N) R_3 = 0$

$\Rightarrow -R_3 I_1 + (R_2 + R_3) I_N = 0$

$\Rightarrow I_1 = \frac{R_2 + R_3}{R_3} I_N$

so, $(r + R_1 + R_3)\left(\frac{R_2 + R_3}{R_3}\right) I_N - R_3 I_N = V_s$

$\Rightarrow \frac{I_N}{R_3}\left[(r + R_1 + R_3) R_2 + (r + R_1) R_3 + R_3^2 - R_3^2\right] = V_s$

$\Rightarrow I_N = \frac{V_s R_3}{(r + R_1 + R_3) R_2 + R_3 (r + R_1)}$ //

(b) → $R_N = R_2 + \frac{R_3 (r + R_1)}{r + R_1 + R_3}$ //

(c) →
Norton's
equiv. ckt.

$$I_L = I_N \cdot \frac{R_N}{R_N + R_L}$$

$$\Rightarrow I_L = I_N \frac{1}{1 + R_L/R_N}$$

$$\Rightarrow I_L = \frac{V_s R_3}{R_2(r+R_1+R_3) + R_3(r+R_1)} \cdot \frac{1}{1 + \frac{R_L(r+R_1+R_3)}{R_2(r+R_1+R_3) + R_3(r+R_1)}}$$

$$\Rightarrow I_L = \frac{V_s R_3 \left[R_2(r+R_1+R_3) + R_3(r+R_1)\right]}{\left[R_2(r+R_1+R_3) + R_3(r+R_1)\right]\left[R_2(r+R_1+R_3) + R_3(r+R_1) + R_L(r+R_1+R_3)\right]}$$

$$\Rightarrow I_L = \frac{V_s R_3}{R_2(r+R_1+R_3) + R_3(r+R_1) + R_L(r+R_1+R_3)}$$

Max. Power Transfer theorem:

max. power delivered from a resistive network to load R_L when R_L is equal to the int. resistance measured looking back into the terminals of the network. This is known as maximum power transfer theorem for a resistive ckt.

$$I_L = \frac{V}{r + R_L}$$

Power in resistance, $P_L = I_L^2 R_L$

Joule heating

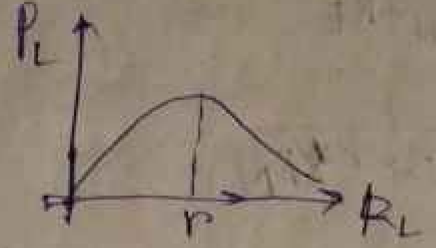
So, $P_L = \frac{V^2}{(r+R_L)^2} R_L$

for max, $\frac{dP_L}{dR_L} = 0 \Rightarrow \frac{1}{(r+R_L)^2} + \frac{R_L(-2)}{(R_L+r)^3} = 0$

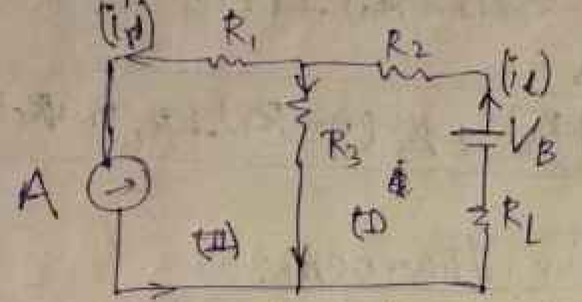
$$\Rightarrow \frac{2R_L}{r+R_L} = 1 \Rightarrow r + R_L = 2R_L$$

$\Rightarrow$ $\boxed{R_L = r}$ condition for max. power absorbed by the load resis^{ce}

i.e. ~~In~~ Load resistance = Source resistance



Reciprocality Theorem: In a ckt containing linear ohmic resistance & energy source the response remain unchanged when the positions of the excitation & response are changed.



position of battery & Ammeter is interchanged.

Using KVL

(I) → $V_B = i_l R_2 + (i_l - i_r') R_3 + i_l R_L$

(II) → $i_r' R_1 + (i_r' - i_l) R_3 = 0$

$\Rightarrow i_l = \frac{R_1 + R_3}{R_3} i_r'$

$V_B = i_l (R_2 + R_3 + R_L) - i_r' R_3$

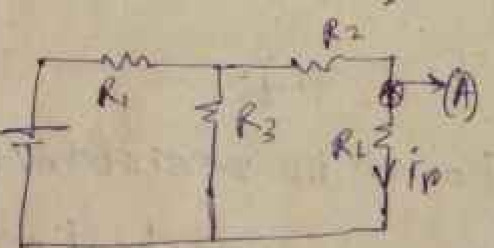
$= \left[\frac{R_1 + R_3}{R_3} (R_2 + R_3 + R_L) - R_3 \right] i_r'$

$\Rightarrow \frac{i_r'}{R_3} \left[(R_2 + R_L)(R_1 + R_3) + R_1 R_3 + R_3^2 - R_3^2 \right] = V_B$

$\Rightarrow i_r' = \frac{V_B R_3}{(R_2 + R_L)(R_1 + R_3) + R_1 R_3}$

$i_r = \frac{R_3 V_B}{(R_L + R_2)(R_1 + R_3) + R_1 R_3}$

(we get it from Thevnin theorem, see before)

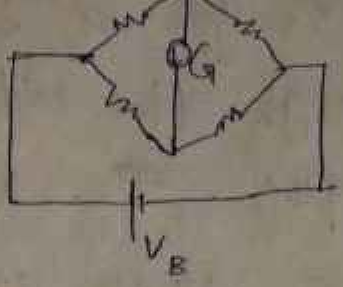


Hence, $\boxed{i_r' = i_r}$

This establish Reciprocality Theorem.

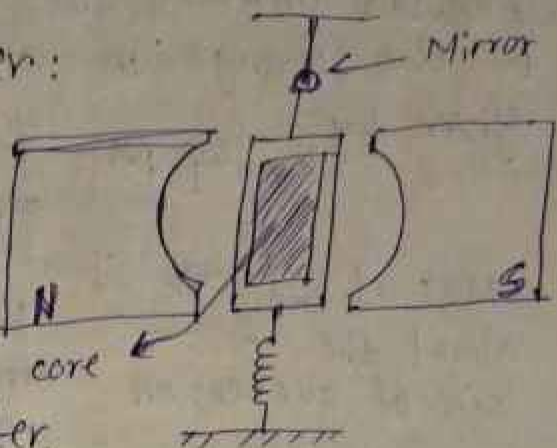
Wheatstone Bridge:

If we interchange V_B & G, bridge will be balanced.



L Ballistic Galvanometer:

Used to measure charge/current.
Ammeter & voltmeter are the modified form of the Galvanometer.



Moving coil suspended type here a movable coil is suspended in the magnetic field of a permanent magnet. When current passes through the coil it experiences a torque & rotate until deflecting torque ~~tt~~ is balanced by the restoring torque of the ~~s~~ fibre.

Angle of deflection gives a measure of current. for steady current measurement, the galvanometer is designed that a steady deflection is obtained without any oscillation. Such a galvanometer is called a dead-beat galvanometer.

Moving coil galvanometer, d Arsonval galvanometer:

Let; N: no. of turns in the coil

A: Area of the coil.

Let $i = \frac{dq}{dt}$ be the instantaneous current through the coil. So,

Torque, $\vec{\tau} = N i \vec{A} \times \vec{B}$

$\tau = NiAB$, as plane of the coil is $\parallel^{el}$ to $\vec{B}$.

Angular impulse given to the coil is equal to the change in angular momentum and is given by, $I\omega_1 = \int_0^t \tau \, dt = NAB \int_0^t i \, dt$

$$\underbrace{I}_{\text{MOI of coil about the axis of suspension}} \underbrace{\omega_1}_{\text{sudden angular vel. acquired by the coil.}} = NABq \quad \text{———} (1)$$

Motion starts from rest at the position $\theta = 0$.

$KE = \frac{1}{2} I\omega_1^2 \Rightarrow$ K.E. of the coil is used to twisting the suspension fibre.

So, $KE = \int_0^{\theta_0} \tau \, d\theta$

$\Rightarrow \frac{1}{2} I\omega_1^2 = c \int_0^{\theta_0} \theta \, d\theta$

$\Rightarrow \frac{1}{2} I\omega_1^2 = c \frac{\theta_0^2}{2}$

$\Rightarrow I\omega_1^2 = c.\theta_0^2 \quad$ — (2)

$\tau = c\theta$, c → couple per unit angle of twist.

Here, no damping is assumed.

Now,

$\tau_{res} = -c\theta \Rightarrow I \frac{d^2\theta}{dt^2} + c\theta = 0$

So $\Rightarrow \frac{d^2\theta}{dt^2} + \left(\frac{c}{I}\right)\theta = 0$

So, $T = 2\pi \sqrt{I/c} \Rightarrow I = \frac{cT^2}{4\pi^2}$ — (3)

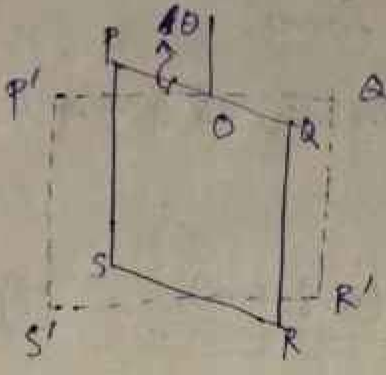
(1), (2) $\rightarrow \omega_1^2 = \left(\frac{NABq}{I}\right)^2 = \frac{c\,\theta_0^2}{I}$

$\Rightarrow (NAB)^2 q^2 = I c \theta_0^2 = \frac{cT^2}{4\pi^2} . c\theta_0^2$

$\Rightarrow q^2 = \dfrac{c^2 T^2 \theta_0^2}{4\pi^2 (NAB)^2}$

$\Rightarrow q = \dfrac{c T \theta_0}{2\pi NAB}$ —— (4)

Thus noting the 1st undamped throw θ_0, it is possible to measure the charge q.



Electromagnetic Damping ~~due to air friction~~ is proportional to the ang. vel. $\frac{d\theta}{dt}$.

Induced emf, $\mathcal{E} = -\dfrac{d\phi_B}{dt} = -\dfrac{d}{dt}\left[2 \cdot PS(OP \cdot d\theta) B\right] N$

$\Rightarrow \mathcal{E} = -\dfrac{d}{dt}(ABN\theta)$ $\quad \Big| \; A = 2 PS \cdot OP$

$\Rightarrow \mathcal{E} = -NAB\left(\dfrac{d\theta}{dt}\right) = -G\dfrac{d\theta}{dt}$

Where, $G = NAB$.

R: resistance of the coil.

current $i = \dfrac{\mathcal{E}}{R}$

opposing torque due to this current is

$-NiAB = -\dfrac{NAB}{R} \cdot \left(-NAB\dfrac{d\theta}{dt}\right) = \dfrac{G^2}{R}\dfrac{d\theta}{dt}$

It's Electromagnetic damping term.

Now, Air Damping $= a\dfrac{d\theta}{dt}$ (let) $\quad (a = \text{const.})$

Eq^n of motion: $I\dfrac{d^2\theta}{dt^2} + \left(a + \dfrac{G^2}{R}\right)\dfrac{d\theta}{dt} + c\theta = 0$

$\Rightarrow \cancel{\dfrac{d^2\theta}{dt^2} + \dfrac{1}{I}\left(a + \dfrac{G^2}{R}\right)} \; \omega.$

$$\frac{d^2\theta}{dt^2} + 2b\frac{d\theta}{dt} + \omega_0^2\theta = 0 \quad \text{—— (5)}$$

Where, $2b = \frac{1}{I}\left(a + \frac{G^2}{R}\right)$ & $\omega_0^2 = \frac{c}{I}$

Eq^n (5) has sol^n of the form,

$$\theta = e^{-bt}\left[A_1 e^{+\sqrt{b^2-\omega_0^2}\,t} + A_2 e^{-\sqrt{b^2-\omega_0^2}}\right] \quad —(6)$$

In Ballistic Galvanometer, $b^2 < \omega_0^2$.

damping is low

So, sol^n will be in the form,

$$\theta = \theta_0 e^{-bt} \sin\left(\sqrt{\omega_0^2 - b^2}\,t + \alpha\right)$$

(α is const)

$$\Rightarrow \underline{\theta = \theta_0 e^{-bt} \sin(\omega t + \alpha)} \quad —(7)$$

$\omega = \sqrt{\omega_0^2 - b^2}$

At $t = 0$, $\theta = 0$. so, $0 = 1 \cdot \sin\alpha$

$\Rightarrow \alpha = 0$

So, $\underline{\theta = \theta_0 e^{-bt} \sin\omega t}$ —(8)

Time period, $T = \frac{2\pi}{\omega} \Rightarrow T = \frac{2\pi}{\sqrt{\omega_0^2 - b^2}}$

Deflection is ~~max~~ max on either side of central position at times given by

$\sin\omega t = \pm 1 \Rightarrow \omega t = (2n+1)\frac{\pi}{2} = \cancel{(n+\frac{1}{2})\frac{\pi}{4}}$

$\Rightarrow t = \cancel{(n+1/2)\frac{\pi}{4\omega}}\ (2n+1) \cdot \left(\frac{2\pi}{\omega}\right) \cdot \frac{1}{4}$

$\Rightarrow t = \frac{T}{4}, \frac{3T}{4}, \frac{5T}{4}, \ldots$

So, $\theta_1 = \theta_0 e^{-\frac{bT}{4}}$, $\theta_2 = \theta_0 e^{-\frac{3T}{4}}$, $\theta_3 = \theta_0 e^{-\frac{5T}{4}}, \ldots$

$\frac{\theta_1}{\theta_2} = e^{-bT/4 + \frac{3T}{4}} = \frac{\theta_2}{\theta_3} = \ldots = \frac{\theta_n}{\theta_{n+1}}$

$\downarrow$ Ratio of the successive amplitudes

$\Rightarrow \frac{\theta_1}{\theta_2} = e^{\frac{bT}{2}} \Rightarrow \underline{d = e^{\frac{bT}{2}}}$

d is decrement .

log decrement, $\underline{\lambda = \ln d}$

$\Rightarrow \lambda = \ln e^{+\frac{bT}{2}} \Rightarrow \underline{\lambda = \frac{bT}{2}}$

So, $\theta_1 = \theta_0 \, e^{-\frac{bT}{4}} \Rightarrow \underline{\theta_0 = \theta_1 \, e^{\lambda/2}}$

$\Rightarrow \underline{\theta_0 = \theta_1 (1 + \frac{\lambda}{2})}$ [If λ is small]

$$\frac{\theta_1}{\theta_{n+1}} = \frac{\theta_1}{\theta_2} \frac{\theta_2}{\theta_3} \cdots \frac{\theta_n}{\theta_{n+1}} = d^n$$

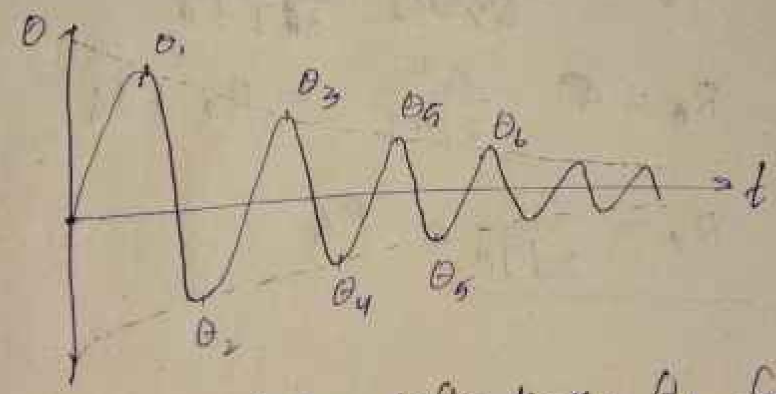
$$\lambda = \ln d \Rightarrow \lambda = \ln \left(\frac{\theta_1}{\theta_{n+1}}\right)^{1/d}$$

$$\Rightarrow \underline{\lambda = \frac{1}{d} \ln \frac{\theta_1}{\theta_{n+1}}}$$

For no damping, $q = \frac{T}{2\pi} \frac{c}{nAB} \theta_1$

For ~~damp~~ damping, $q = \frac{T}{2\pi} \cdot \frac{c}{nAB} \cdot \theta_0$

$\Rightarrow \underline{q = \frac{T}{2\pi} \cdot \frac{c}{nAB} \, \theta_1 (1 + \frac{\lambda}{2})}$ — (9)



~~for~~ Steady state deflection, θ_d for a steady current i_d, the deflecting torque $NABi_d$ is ~~res~~ equal to restoring torque, $c\theta_d$. $NABi_d = c\theta_d$

$$\Rightarrow \frac{c}{NAB} = \left(\frac{i_d}{\theta_d}\right)$$

By this we can find the const. $\frac{c}{NAB}$

Critical Damping ~~Res~~ Resistance (CDR):

The damping const. $2b = \frac{1}{I}\left(a + \frac{G^2}{R}\right)$ where

~~R is~~ $R = R_G + R_{ext}$.

of galvanometer

R_c is called external critical damping resistance. (CDR).

R_{ext} should be chosen greater than certain critical value R_c in order to have oscillatory motion

It is ~~given~~ given by the condition,

$$\cancel{d = \omega_0} \quad \cancel{b =} \quad b = \omega_0$$

$$\Rightarrow \frac{1}{2I}\left(a + \frac{G^2}{R}\right) = \sqrt{\frac{C}{I}}$$

$$\Rightarrow \left(\overset{\to 0}{\frac{a}{2I}} - \sqrt{\frac{C}{I}}\right) = -\frac{G^2}{2RI} \qquad \Big| \; a << \frac{G^2}{R}$$

$$\Rightarrow R = \cancel{\frac{2I}{G^2}\sqrt{\frac{C}{I}}} \; \frac{G^2}{2I}\sqrt{\frac{I}{C}}$$

$$\Rightarrow R = \frac{G^2}{2\sqrt{CI}} = R_c + R_G$$

$$\Rightarrow R_c = \frac{G^2}{2\sqrt{IC}} - R_G$$